

Imprecise Continuous-Time Markov Chains: Efficient Computational Methods with Guaranteed Error Bounds

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Abstract

Imprecise continuous-time Markov chains are a robust type of continuous-time Markov chains that allow for partially specified time-dependent parameters. Computing inferences for them requires the solution of a non-linear differential equation. As there is no general analytical expression for this solution, efficient numerical approximation methods are essential to the applicability of this model. We here improve the uniform approximation method of [Krak et al. \(2016\)](#) in two ways and propose a novel and more efficient adaptive approximation method. For ergodic chains, we also provide a method that allows us to approximate stationary distributions up to any desired maximal error.

Keywords: Imprecise continuous-time Markov chain; lower transition operator; lower transition rate operator; approximation method; ergodicity; coefficient of ergodicity.

1. Introduction

Markov chains are a popular type of stochastic processes that can be used to model a variety of systems with uncertain dynamics, both in discrete and continuous time. In many applications, however, the core assumption of a Markov chain—i.e., the Markov property—is not entirely justified. Moreover, it is often difficult to exactly determine the parameters that characterise the Markov chain. In an effort to handle these modelling errors in an elegant manner, several authors have recently turned to imprecise probabilities ([de Cooman et al., 2009](#); [Škulj and Hable, 2013](#); [Hermans and de Cooman, 2012](#); [Škulj, 2015](#); [Krak et al., 2016](#); [De Bock, 2017](#)).

As [Krak et al. \(2016\)](#) thoroughly demonstrate, making inferences about an imprecise continuous-time Markov chain—determining lower and upper expectations or probabilities—requires the solution of a non-linear vector differential equation. To the best of our knowledge, this differential equation cannot be solved analytically, at least not in general. [Krak et al. \(2016\)](#) proposed a method to numerically approximate the solution of the differential equation, and argued that it outperforms the approximation method that [Škulj \(2015\)](#) previously introduced. One of the main results of this contribution is a novel approximation method that outperforms that of [Krak et al. \(2016\)](#).

An important property—both theoretically and practically—of continuous-time Markov chains is the behaviour of the solution of the differential equation as the time parameter recedes to infinity. If regardless of the initial condition the solution converges, we say that the chain is ergodic. We show that in this case the approximation is guaranteed to converge as well. This constitutes the second main result of this contribution and serves as a motivation behind the novel approximation method. Furthermore, we also quantify a worst-case convergence rate for the approximation. This unites the work of [Škulj \(2015\)](#), who studied the rate of convergence for discrete-time Markov chains,

and De Bock (2017), who studied the ergodic behaviour of continuous-time Markov chains from a qualitative point of view. One of the uses of our worst-case convergence rate is that it allows us to approximate the limit value of the solution up to a guaranteed error.

To ensure the readability of the main text, we have gathered the proofs of all the results in the Appendix. In this Appendix, we also discuss the ergodicity of both discrete and continuous-time Markov chains more thoroughly.

2. Mathematical preliminaries

Throughout this contribution, we denote the set of real, non-negative real and strictly positive real numbers by \mathbb{R} , $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{> 0}$, respectively. The set of natural numbers is denoted by \mathbb{N} , if we include zero we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any set S , we let $|S|$ denote its cardinality. If a and b are two real numbers, we say that a is lower (greater) than b if $a \leq b$ ($a \geq b$), and that a is strictly lower (greater) than b if $a < b$ ($a > b$).

2.1 Gambles and norms

We consider a finite *state space* \mathcal{X} , and are mainly concerned with real-valued functions on \mathcal{X} . All of these real-valued functions on \mathcal{X} are collected in the set $\mathcal{L}(\mathcal{X})$, which is a vector space. If we identify the state space \mathcal{X} with $\{1, \dots, |\mathcal{X}|\}$, then any function $f \in \mathcal{L}(\mathcal{X})$ can be identified with a vector: for all $x \in \mathcal{X}$, the x -component of this vector is $f(x)$. A special function on \mathcal{X} is the indicator \mathbb{I}_A of an event A . For any $A \subseteq \mathcal{X}$, it is defined for all $x \in \mathcal{X}$ as $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise. In order not to obfuscate the notation too much, for any $y \in \mathcal{X}$ we write \mathbb{I}_y instead of $\mathbb{I}_{\{y\}}$. If it is required from the context, we will also identify the real number $\gamma \in \mathbb{R}$ with the map γ from \mathcal{X} to \mathbb{R} , defined as $\gamma(x) = \gamma$ for all $x \in \mathcal{X}$.

We provide the set $\mathcal{L}(\mathcal{X})$ of functions with the standard maximum norm $\|\cdot\|$, defined for all $f \in \mathcal{L}(\mathcal{X})$ as $\|f\| := \max \{|f(x)| : x \in \mathcal{X}\}$. A seminorm that captures the variation of $f \in \mathcal{L}(\mathcal{X})$ will also be of use; we therefore define the variation seminorm $\|f\|_v := \max f - \min f$. Since the value $\|f\|_v / 2$ occurs often in formulas, we introduce the shorthand notation $\|f\|_c := \|f\|_v / 2$.

2.2 Non-negatively homogeneous operators

An operator A that maps $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ is *non-negatively homogeneous* if for all $\mu \in \mathbb{R}_{\geq 0}$ and all $f \in \mathcal{L}(\mathcal{X})$, $A(\mu f) = \mu A f$. The maximum norm $\|\cdot\|$ for functions induces an operator norm:

$$\|A\| := \sup\{\|A f\| : f \in \mathcal{L}(\mathcal{X}), \|f\| = 1\}.$$

If for all $\mu \in \mathbb{R}$ and all $f, g \in \mathcal{L}(\mathcal{X})$, $A(\mu f + g) = \mu A f + A g$, then the operator A is *linear*. In that case, it can be identified with a matrix of dimension $|\mathcal{X}| \times |\mathcal{X}|$, the (x, y) -component of which is $[A \mathbb{I}_y](x)$. The identity operator I is an important special case, defined for all $f \in \mathcal{L}(\mathcal{X})$ as $I f := f$.

Two types of non-negatively homogeneous operators play a vital role in the theory of imprecise Markov chains: lower transition operators and lower transition rate operators.

Definition 1. An operator \underline{T} from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ is called a *lower transition operator* if for all $f \in \mathcal{L}(\mathcal{X})$ and all $\mu \in \mathbb{R}_{\geq 0}$:

$$L1: \underline{T}f \geq \min f; \quad L2: \underline{T}(f + g) \geq \underline{T}f + \underline{T}g; \quad L3: \underline{T}(\mu f) = \mu \underline{T}f.$$

Every lower transition operator \underline{T} has a conjugate upper transition operator \overline{T} , defined for all $f \in \mathcal{L}(\mathcal{X})$ as $\overline{T}f := -\underline{T}(-f)$.

Definition 2. An operator \underline{Q} from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ is called a lower transition rate operator if for any $f, g \in \mathcal{L}(\mathcal{X})$, any $\mu \in \mathbb{R}_{\geq 0}$, any $\gamma \in \mathbb{R}$ and any $x, y \in \mathcal{X}$ such that $x \neq y$:

$$\begin{aligned} R1: \underline{Q}\gamma &= 0; & R3: \underline{Q}(\mu f) &= \mu \underline{Q}f; \\ R2: \underline{Q}(f + g) &\geq \underline{Q}f + \underline{Q}g; & R4: [\underline{Q}\mathbb{I}_x](y) &\geq 0. \end{aligned}$$

The conjugate lower transition rate operator \overline{Q} is defined for all $f \in \mathcal{L}(\mathcal{X})$ as $\overline{Q}f := -\underline{Q}(-f)$.

As will become clear in Section 3, lower transition operators and lower transition rate operators are tightly linked. For instance, we can use a lower transition rate operator to construct a lower transition operator. One way is to use Eqn. (1) further on. Another one is given in the following proposition, which is a strengthened version of (De Bock, 2017, Proposition 5).

Proposition 3. Consider any lower transition rate operator \underline{Q} and any $\delta \in \mathbb{R}_{\geq 0}$. Then the operator $(I + \delta \underline{Q})$ is a lower transition operator if and only if $\delta \|\underline{Q}\| \leq 2$.

We end this section with the first—although minor—novel result of this contribution. The norm of a lower transition rate operator is essential for all the approximation methods that we will discuss. The following proposition supplies us with an easy formula for determining it.

Proposition 4. Let \underline{Q} be a lower transition rate operator. Then $\|\underline{Q}\| = 2 \max\{|\underline{Q}\mathbb{I}_x](x)| : x \in \mathcal{X}\}$.

Example 1. Consider a binary state space $\mathcal{X} = \{0, 1\}$ and two closed intervals $[\underline{q}_0, \overline{q}_0] \subset \mathbb{R}_{\geq 0}$ and $[\underline{q}_1, \overline{q}_1] \subset \mathbb{R}_{\geq 0}$. Let

$$\underline{Q}f := \min \left\{ \begin{bmatrix} q_0(f(1) - f(0)) \\ q_1(f(0) - f(1)) \end{bmatrix} : q_0 \in [\underline{q}_0, \overline{q}_0], q_1 \in [\underline{q}_1, \overline{q}_1] \right\} \text{ for all } f \in \mathcal{L}(\mathcal{X}).$$

Then one can easily verify that \underline{Q} is a lower transition rate operator.

Krak et al. (2016) also consider a running example with a binary state space, but they let $\mathcal{X} := \{\text{healthy}, \text{sick}\}$. We here identify *healthy* with 0 and *sick* with 1. In (Krak et al., 2016, Example 18), they propose the following values for the transition rates: $[\underline{q}_0, \overline{q}_0] := [1/52, 3/52]$ and $[\underline{q}_1, \overline{q}_1] := [1/2, 2]$. It takes Krak et al. a lot of work to determine the exact value of the norm of \underline{Q} , see (Krak et al., 2016, Example 19). We simply use Proposition 4: $\|\underline{Q}\| = 2 \max\{3/52, 2\} = 4$.

3. Imprecise continuous-time Markov chains

For any lower transition rate operator \underline{Q} and any $f \in \mathcal{L}(\mathcal{X})$, Škulj (2015) has shown that the differential equation

$$\frac{d}{dt} \underline{T}_t f = \underline{Q} \underline{T}_t f. \quad (1)$$

with initial condition $\underline{T}_0 f := f$ has a unique solution for all $t \in \mathbb{R}_{\geq 0}$. Later, De Bock (2017) proved that the time-dependent operator \underline{T}_t itself satisfies a similar differential equation, and that it is a

lower transition operator. Finding the unique solution of Eqn. (1) is non-trivial. Fortunately, we can approximate this solution, as by (De Bock, 2017, Proposition 10)

$$\underline{T}_t = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} Q \right)^n. \quad (2)$$

Example 2. In the simple case of Example 1, we can use Eqn. (2) to obtain analytical expressions for the solution of Eqn. (1). Assume that $q_0 + \bar{q}_1 > 0$ and fix some $t \in \mathbb{R}_{\geq 0}$. Then

$$[\underline{T}_t f](0) = f(0) + q_0 h(t) \text{ and } [\underline{T}_t f](1) = f(1) - \bar{q}_1 h(t) \text{ for all } f \in \mathcal{L}(\mathcal{X}) \text{ with } f(0) \leq f(1),$$

where $h(t) := \|f\|_v (q_0 + \bar{q}_1)^{-1} (1 - e^{-t(q_0 + \bar{q}_1)})$. The case $f(0) \geq f(1)$ yields similar expressions.

For a linear lower transition rate operator \underline{Q} —i.e., if it is a transition rate matrix Q —Eqn. (2) reduces to the definition of the matrix exponential. It is well-known—see (Anderson, 1991)—that this matrix exponential $T_t = e^{tQ}$ can be interpreted as the transition matrix at time t of a time-homogeneous or stationary continuous-time Markov chain: the (x, y) -component of T_t is the probability of being in state y at time t if the chain started in state x at time 0. Therefore, it follows that the expectation of the function $f \in \mathcal{L}(\mathcal{X})$ at time $t \in \mathbb{R}_{\geq 0}$ conditional on the initial state $x \in \mathcal{X}$, denoted by $E(f(X_t)|X_0 = x)$, is equal to $[\underline{T}_t f](x)$.

As Eqn. (2) is a non-linear generalisation of the definition of the matrix exponential, we can interpret \underline{T}_t as the non-linear generalisation of the matrix exponential $T_t = e^{tQ}$. Extending this parallel, we might interpret \underline{T}_t as the non-linear generalisation of the transition matrix—i.e., as the lower transition operator—at time t of a generalised continuous-time Markov chain. In fact, Krak et al. (2016) prove that this is indeed the case. They show that—under some conditions on \underline{Q} — $[\underline{T}_t f](x)$ can be interpreted as the tightest lower bound for $E(f(X_t)|X_0 = x)$ with respect to a set of—not necessarily Markovian—stochastic processes that are consistent with \underline{Q} . Krak et al. (2016) argue that, just like a transition rate matrix Q characterises a (precise) continuous-time Markov chain, a lower transition rate operator \underline{Q} characterises a so-called imprecise continuous-time Markov chain.

The main objective of this contribution is to determine $\underline{T}_t f$ for some $f \in \mathcal{L}(\mathcal{X})$ and some $t \in \mathbb{R}_{> 0}$. Our motivation is that, from an applied point of view on imprecise continuous-time Markov chains, what one is most interested in are tight lower and upper bounds on expectations of the form $E(f(X_t)|X_0 = x)$. As explained above, the lower bound is given by $\underline{E}(f(X_t)|X_0 = x) = [\underline{T}_t f](x)$. Similarly, the upper bound is given by $\bar{E}(f(X_t)|X_0 = x) = -[\underline{T}_t(-f)](x)$. Note that the lower (or upper) probability of an event $A \subseteq \mathcal{X}$ conditional on the initial state x is a special case of a lower (or upper) expectation: $\underline{P}(X_t \in A|X_0 = x) = \underline{E}(\mathbb{I}_A(X_t)|X_0 = x)$ and similarly for the upper probability. Hence, for the sake of generality we can focus on $\underline{T}_t f$ and forget about its interpretation. As in most cases analytically solving Eqn. (1) is infeasible or even impossible, we resort to methods that yield an approximation up to some guaranteed maximal error.

4. Approximation methods

Škulj (2015) was, to the best of our knowledge, the first to propose methods that approximate the solution $\underline{T}_t f$ of Eqn. (1). He proposes three methods: one with a uniform grid, a second with an adaptive grid and a third that is a combination of the previous two. In essence, he determines a step size δ and then approximates $\underline{T}_{t+\delta} f$ with $e^{\delta Q} \underline{T}_t f$, where Q is a transition rate matrix determined

from \underline{Q} and $\underline{T}_t f$. One drawback of this method is that it needs the matrix exponential $e^{\delta \underline{Q}}$, which—in general—needs to be approximated as well. Škulj (2015) mentions that his methods turn out to be quite computationally heavy, even if the uniform and adaptive methods are combined.

We consider two alternative approximation methods—one with a uniform grid and one with an adaptive grid—that both work in the same way. First, we pick a small step $\delta_1 \in \mathbb{R}_{\geq 0}$ and apply the operator $(I + \delta_1 \underline{Q})$ to the function $g_0 = f$, resulting in a function $g_1 := (I + \delta_1 \underline{Q})f$. Recall from Proposition 3 that if we want $(I + \delta_1 \underline{Q})$ to be a lower transition operator, then we need to demand that $\delta_1 \|\underline{Q}\| \leq 2$. Next, we pick a (possibly different) small step $\delta_2 \in \mathbb{R}_{\geq 0}$ such that $\delta_2 \|\underline{Q}\| \leq 2$ and apply the lower transition operator $(I + \delta_2 \underline{Q})$ to the function g_1 , resulting in a function $g_2 := (I + \delta_2 \underline{Q})g_1$. If we continue this process until the sum of all the small steps is equal to t , then we end up with an approximation for $\underline{T}_t f$. More formally, let $s := (\delta_1, \dots, \delta_k)$ denote a sequence in $\mathbb{R}_{\geq 0}$ such that, for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| \leq 2$. Using this sequence s we define the *approximating lower transition operator*

$$\Phi(s) := (I + \delta_k \underline{Q}) \cdots (I + \delta_1 \underline{Q}).$$

What we are looking for is a convenient way to determine the sequence s such that the error $\|\underline{T}_t f - \Phi(s)f\|$ is guaranteed to be lower than some desired maximal error $\epsilon \in \mathbb{R}_{>0}$.

4.1 Using a uniform grid

Krak et al. (2016) provide one way to determine the sequence s . They assume a uniform grid, in the sense that all elements of the sequence s are equal to δ . The step size δ is completely determined by the desired maximal error ϵ , the time t , the variation norm of the function f and the norm of \underline{Q} ; (Krak et al., 2016, Proposition 8.5) guarantees that the actual error is lower than ϵ . Algorithm 1 provides a slightly improved version of (Krak et al., 2016, Algorithm 1). The improvement is due to Proposition 3: we demand that $n \geq t \|\underline{Q}\| / 2$ instead of $n \geq t \|\underline{Q}\|$.

Algorithm 1: Uniform approximation

Data: A lower transition rate operator \underline{Q} , a function $f \in \mathcal{L}(\mathcal{X})$, a maximal error $\epsilon \in \mathbb{R}_{>0}$, and a time point $t \in \mathbb{R}_{\geq 0}$.

Result: $\underline{T}_t f \pm \epsilon$

```

1  $g_0 \leftarrow f$ 
2 if  $\|f\|_c = 0$  or  $\|\underline{Q}\| = 0$  or  $t = 0$  then  $(n, \delta) \leftarrow (0, 0)$ 
3 else
4    $n \leftarrow \lceil \max\{t \|\underline{Q}\| / 2, t^2 \|\underline{Q}\|^2 \|f\|_c / \epsilon\} \rceil$ 
5    $\delta \leftarrow t/n$ 
6   for  $i = 0, \dots, n-1$  do
7      $g_{i+1} \leftarrow g_i + \delta \underline{Q} g_i$ 
8 return  $g_n$ 
```

More formally, for any $t \in \mathbb{R}_{\geq 0}$ and any $n \in \mathbb{N}$ such that $t \|\underline{Q}\| \leq 2n$, we consider the *uniformly approximating lower transition operator*

$$\Psi_t(n) := \left(I + \frac{t}{n} \underline{Q} \right)^n.$$

Table 1: Comparison of the presented approximation methods, obtained using a naive, unoptimised implementation of the algorithms in Python. N is the total number of iterations, D_ϵ ($D_{\epsilon'}$) is the average duration—in seconds, averaged over 50 independent runs—without (with) keeping track of ϵ' , and ϵ_a is the actual error. The Python code is made available at github.com/alexander-e/ictmc.

Method	N	D_ϵ	$D_{\epsilon'}$	$\epsilon' \times 10^3$	$\epsilon_a \times 10^3$
Uniform	8,000	0.0345	0.0574	0.430	0.0335
Uniform	250	0.00171	0.0264	13.8	1.07
Adaptive with $m = 1$	3,437	0.0371	0.0428	1.000	0.108
Adaptive with $m = 20$	3,456	0.0143	0.0254	0.992	0.107
Uniform ergodic with $m = 1$	6,133	0.0264	0.0449	0.560	0.0437

As a special case, we define $\Psi_t(0) := I$. The following theorem then guarantees that the choice of n in Algorithm 1 results in an error $\|\underline{T}_t f - \Psi_t(n)f\|$ that is lower than the desired maximal error ϵ .

Theorem 5. *Let \underline{Q} be a lower transition rate operator and fix some $f \in \mathcal{L}(\mathcal{X})$, $t \in \mathbb{R}_{\geq 0}$ and $\epsilon \in \mathbb{R}_{>0}$. If we use Algorithm 1 to determine n , δ and g_0, \dots, g_n , then we are guaranteed that*

$$\|\underline{T}_t f - \Psi_t(n)f\| = \|\underline{T}_t f - g_n\| \leq \epsilon' := \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{n-1} \|g_i\|_c \leq \epsilon.$$

Theorem 5 is an extension of (Krak et al., 2016, Proposition 8.5). We already mentioned that the demand $n \geq t \|\underline{Q}\|$ can be relaxed to $n \geq t \|\underline{Q}\|/2$. Furthermore, it turns out that we can compute an upper bound ϵ' on the error that is (possibly) lower than the desired maximal error ϵ . If we want to determine this ϵ' while running Algorithm 1, we simply need to add $\epsilon' \leftarrow 0$ to line 1 and insert $\epsilon' \leftarrow \epsilon' + \delta^2 \|\underline{Q}\|^2 \|g_i\|_c$ just before line 7.

Example 3. We again consider the simple case of Example 1 and illustrate the use of Theorem 5 with a numerical example based on (Krak et al., 2016, Example 20). Krak et al. (2016) use Algorithm 1 to approximate $\underline{T}_1 \mathbb{I}_1$, and find that $n = 8,000$ guarantees an error lower than the desired maximal error $\epsilon := 1 \times 10^{-3}$. As reported in Table 1, we use Theorem 5 to compute ϵ' . We find that $\epsilon' \approx 0.430 \times 10^{-3}$, which is approximately a factor two smaller than the desired maximal error ϵ .

In this case, since we know the analytical expression for $\underline{T}_1 \mathbb{I}_1$ from Example 2, we can determine the actual error $\epsilon_a = \|\underline{T}_1 \mathbb{I}_1 - \Psi_1(8000) \mathbb{I}_1\|$. Quite remarkably, the actual error is approximately 3.35×10^{-5} , which is roughly 30 times smaller than the desired maximal error. This leads us to think that the number of iterations used by the uniform method is too high. In fact, we find that using as few as 250 iterations—roughly $8,000/30$ —already results in an actual error that is approximately equal to the desired one: $\|\underline{T}_1 \mathbb{I}_1 - \Psi_1(250) \mathbb{I}_1\| \approx 1.07 \times 10^{-3}$.

4.2 Using an adaptive grid

In Example 3, we noticed that the maximal desired error was already satisfied for a uniform grid that was much coarser than that constructed by Algorithm 1. Because of this, we are led to believe that we can find a better approximation method than the uniform method of Algorithm 1.

To this end, we now consider grids where, for some integer m , every m consecutive time steps in the grid are equal. In particular, we consider a sequence $\delta_1, \dots, \delta_n$ in $\mathbb{R}_{\geq 0}$ and some $k \in \mathbb{N}$ such that $1 \leq k \leq m$ and, for all $i \in \{1, \dots, n\}$, $\delta_i \|Q\| \leq 2$. From such a sequence, we then construct the m -fold approximating lower transition operator:

$$\Phi_{m,k}(\delta_1, \dots, \delta_n) := (I + \delta_n \underline{Q})^k (I + \delta_{n-1} \underline{Q})^m \cdots (I + \delta_1 \underline{Q})^m,$$

where if $n = 1$ only $(I + \delta_1 \underline{Q})^k$ remains and if $n = 2$ only $(I + \delta_2 \underline{Q})^k (I + \delta_1 \underline{Q})^m$ remains.

The uniform approximation method of before is a special case of the m -fold approximating lower transition operator; a more interesting method to construct an m -fold approximation is Algorithm 2. In this algorithm, we re-evaluate the time step every m iterations, possibly increasing its length.

Algorithm 2: Adaptive approximation

Data: A lower transition rate operator Q , a gamble $f \in \mathcal{L}(\mathcal{X})$, an integer $m \in \mathbb{N}$, a tolerance $\epsilon \in \mathbb{R}_{>0}$, and a time period $t \in \mathbb{R}_{\geq 0}$.

Result: $\underline{T}_t f \pm \epsilon$

```

1  $(g_{(0,m)}, \Delta, i) \leftarrow (f, t, 0)$ 
2 if  $\|f\|_c = 0$  or  $\|Q\| = 0$  or  $t = 0$  then  $(n, k) \leftarrow (0, m)$ 
3 else
4   while  $\Delta > 0$  and  $\|g_{(i,m)}\|_c > 0$  do
5      $i \leftarrow i + 1$ 
6      $\delta_i \leftarrow \min\{\Delta, 2/\|Q\|, \epsilon/(t\|Q\|^2\|g_{(i-1,m)}\|_c)\}$ 
7     if  $m\delta_i > \Delta$  then
8        $k_i \leftarrow \lceil \Delta/\delta_i \rceil$ 
9        $\delta_i \leftarrow \Delta/k_i$ 
10    else  $k_i \leftarrow m$ 
11     $g_{(i,0)} \leftarrow g_{(i-1,m)}, \Delta \leftarrow \Delta - k_i \delta_i$ 
12    for  $j = 0, \dots, k_i - 1$  do
13       $g_{(i,j+1)} \leftarrow g_{(i,j)} + \delta_i \underline{Q} g_{(i,j)}$ 
14     $(n, k) \leftarrow (i, k_i)$ 
15 return  $g_{(n,k)}$ 

```

From the properties of lower transition operators, it follows that for all $i \in \{2, \dots, n-1\}$, $\|g_{(i-1,m)}\|_c \leq \|g_{(i-2,m)}\|_c$. Hence, the re-evaluated step size δ_i is indeed larger than (or equal to) the previous step size δ_{i-1} . The only exception to this is the final step size δ_n : it might be that the remaining time Δ is smaller than $m\delta_n$, in which case we need to choose k and δ_n such that $k\delta_n = \Delta$.

Theorem 6 guarantees that the adaptive approximation of Algorithm 2 indeed results in an actual error lower than the desired maximal error ϵ . Even more, it provides a method to compute an upper bound ϵ' of the actual error that is lower than the desired maximal error. Finally, it also states that

the adaptive method of Algorithm 2 needs at most an equal number of iterations than the uniform method of Algorithm 1.

Theorem 6. *Let \underline{Q} be a lower transition rate operator, $f \in \mathcal{L}(\mathcal{X})$, $t \in \mathbb{R}_{\geq 0}$, $\epsilon \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}$. We use Algorithm 2 to determine n and k , and if applicable also k_i , δ_i and $g_{(i,j)}$. If $\|f\|_c = 0$, $\|\underline{Q}\| = 0$ or $t = 0$, then $\|\underline{T}_t f - g_{(n,k)}\| = 0$. Otherwise, we are guaranteed that*

$$\|\underline{T}_t f - \Phi_{m,k}(\delta_1 \dots, \delta_n) f\| = \|\underline{T}_t f - g_{(n,k)}\| \leq \epsilon' := \sum_{i=1}^n \delta_i^2 \|\underline{Q}\|^2 \sum_{j=0}^{k_i-1} \|g_{(i,j)}\|_c \leq \epsilon$$

and that the total number of iterations has an upper bound:

$$\sum_{i=1}^n k_i = (n-1)m + k \leq \left\lceil \max \left\{ \|\underline{Q}\| t/2, t^2 \|\underline{Q}\|^2 \|f\|_c / \epsilon \right\} \right\rceil.$$

Again, we can determine ϵ' while running Algorithm 2. An alternate—less tight—version of ϵ' can be obtained by replacing the sum of $\|g_{(i,j)}\|_c$ for j from 0 to $k_i - 1$ by $k_i \|g_{(i,0)}\|_c = k_i \|g_{(i-1,m)}\|_c$. Determining this alternative ϵ' while running Algorithm 2 adds negligible computational overhead compared to the ϵ' of Theorem 6, as $\|g_{(i-1,m)}\|_c$ is needed to re-evaluate the step size anyway.

The reason why we only re-evaluate the step size δ after every m iterations is twofold. First and foremost, all we currently know for sure is that for all $\delta \in \mathbb{R}_{\geq 0}$ such that $\delta \|\underline{Q}\| \leq 2$, all $m \in \mathbb{N}$ and all $f \in \mathcal{L}(\mathcal{X})$, $\|(I + \delta \underline{Q})^m f\|_c \leq \|f\|_c$. Re-evaluating the step size every m iterations is therefore only justified if a priori we are certain that $\|(I + \delta_i \underline{Q})^m g_{(i-1,m)}\|_c < \|g_{(i-1,m)}\|_c$. We come back to this in Section 5. A second reason is that there might be a trade-off between the time it takes to re-evaluate the step size and the time that is gained by the resulting reduction of the number of iterations. The following numerical example illustrates this trade off.

Example 4. Recall that in Example 3 we wanted to approximate $\underline{T}_1 \mathbb{1}_1$ up to a maximal desired error $\epsilon = 1 \times 10^{-3}$. Instead of using the uniform method of Algorithm 1, we now use the adaptive method of Algorithm 2 with $m = 1$. The initial step size is the same as that of the uniform method, but because we re-evaluate the step size we only need 3,437 iterations, as reported in Table 1. We also find that in this case $\epsilon' = 1.00 \times 10^{-3}$, which is a coincidence. Nevertheless, the actual error of the approximation is 0.108×10^{-3} , which is about ten times smaller than what we were aiming for.

However, fewer iterations do not necessarily imply a shorter duration of the computations. Qualitatively, we can conclude the following from Table 1. First, keeping track of ϵ' increases the duration, as expected. Second, the adaptive method is faster than the uniform method, at least if we choose m large enough. And third, both methods yield an actual error that is at least an order of magnitude lower than the desired maximal error.

5. Ergodicity

Let $\Phi_{m,k}(\delta_1, \dots, \delta_n) f$ be an approximation constructed using the adaptive method of Algorithm 2. Re-evaluating the step size is then only justified if a priori we are sure that

$$1/2 \|(I + \delta_i \underline{Q})^m \Phi_{i-1} f\|_v = \|g_{(i,m)}\|_c < \|g_{(i-1,m)}\|_c = 1/2 \|\Phi_{i-1} f\|_v \text{ for all } i \in \{1, \dots, n-1\},$$

where $\Phi_0 := I$ and $\Phi_i := (I + \delta_i \underline{Q})^m \Phi_{i-1}$. As $(\Phi_{i-1} f) \in \mathcal{L}(\mathcal{X})$, this is definitely true if we require that

$$(\forall \delta \in \{\delta_1, \dots, \delta_{n-1}\})(\forall f \in \mathcal{L}(\mathcal{X})) \|(I + \delta \underline{Q})^m f\|_v < \|f\|_v. \quad (3)$$

In fact, since this inequality is invariant under translation or positive scaling of f , it suffices if

$$(\forall \delta \in \{\delta_1, \dots, \delta_{n-1}\})(\forall f \in \mathcal{L}(\mathcal{X}): 0 \leq f \leq 1) \|(I + \delta \underline{Q})^m f\|_v < 1.$$

Readers that are familiar with (the ergodicity of) imprecise discrete-time Markov chains—see (Hermans and de Cooman, 2012) or (Škulj and Hable, 2013)—will probably recognise this condition, as it states that the (weak) coefficient of ergodicity of $(I + \delta \underline{Q})^m$ should be strictly smaller than 1. For all lower transition operators \underline{T} , Škulj and Hable (2013) define this (weak) *coefficient of ergodicity* as

$$\rho(\underline{T}) := \max \{ \|\underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \}. \quad (4)$$

5.1 Ergodicity of lower transition rate operators

As will become apparent, whether or not combinations of $m \in \mathbb{N}$ and $\delta \in \mathbb{R}_{\geq 0}$ exist such that $\delta \|\underline{Q}\| \leq 2$ and $\rho((I + \delta \underline{Q})^m) < 1$ is tightly connected with the behaviour of $\underline{T}_t f$ for large t . De Bock (2017) proved that for all lower transition rate operator \underline{Q} and all $f \in \mathcal{L}(\mathcal{X})$, the limit $\lim_{t \rightarrow \infty} \underline{T}_t f$ exists. An important case is when this limit is a constant function for all f .

Definition 7 (Definition 2 of (De Bock, 2017)). *The lower transition rate operator \underline{Q} is ergodic if for all $f \in \mathcal{L}(\mathcal{X})$, $\lim_{t \rightarrow \infty} \underline{T}_t f$ exists and is a constant function.*

As shown by De Bock (2017), ergodicity is easily verified in practice: it is completely determined by the signs of $[\overline{Q}\mathbb{I}_x](y)$ and $[\underline{Q}\mathbb{I}_A](z)$, for all $x, y \in \mathcal{X}$ and certain combinations of $z \in \mathcal{X}$ and $A \subset \mathcal{X}$. It turns out that an ergodic lower transition rate operator \underline{Q} does not only induce a lower transition operator \underline{T}_t that converges, it also induces discrete approximations—of the form $(I + \delta_k \underline{Q}) \cdots (I + \delta_1 \underline{Q})$ —with special properties. The following theorem, which we consider to be one of the main results of this contribution, highlights this.

Theorem 8. *The lower transition rate operator \underline{Q} is ergodic if and only if there is some $n < |\mathcal{X}|$ such that $\rho(\Phi(\delta_1, \dots, \delta_k)) < 1$ for one (and then all) $k \geq n$ and one (and then all) sequence(s) $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{>0}$ such that $\delta_i \|\underline{Q}\| < 2$ for all $i \in \{1, \dots, k\}$.*

5.2 Ergodicity and the uniform approximation method

Theorem 8 guarantees that the conditions that were discussed at the beginning of this section are satisfied. In particular, if the lower transition rate operator is ergodic, then there is some $n < |\mathcal{X}|$ such that $\rho((I + \delta \underline{Q})^m) < 1$ for all $m \geq n$ and all $\delta \in \mathbb{R}_{>0}$ such that $\delta \|\underline{Q}\| < 2$. Consequently, if we choose $m \geq |\mathcal{X}| - 1$ then re-evaluating the step size δ will—except maybe for the last re-evaluation—result in a new step size that is strictly greater than the previous one. Therefore, we conclude that if the lower transition rate operator is ergodic, then using the adaptive method of Algorithm 2 is certainly justified; it will result in fewer iterations, provided we choose a large enough m .

Another nice consequence of the ergodicity of a lower transition rate operator \underline{Q} is that we can prove an alternate a priori guaranteed upper bound for the error of uniform approximations.

Proposition 9. *Let \underline{Q} be a lower transition rate operator and fix some $f \in \mathcal{L}(\mathcal{X})$, $m, n \in \mathbb{N}$ and $\delta \in \mathbb{R}_{>0}$ such that $\delta \|\underline{Q}\| < 2$. If $\beta := \rho((I + \delta \underline{Q})^m) < 1$, then*

$$\|\underline{T}_t f - \Psi_t(n)\| \leq \epsilon_e := m\delta^2 \|\underline{Q}\|^2 \|f\|_c \frac{1 - \beta^k}{1 - \beta} \leq \epsilon_d := \frac{m\delta^2 \|\underline{Q}\|^2 \|f\|_c}{1 - \beta},$$

where $t := n\delta$ and $k := \lceil n/m \rceil$. The same is true for $\beta = \rho(\underline{T}_{m\delta})$.

Interestingly enough, the upper bound ϵ_d is not dependent on t (or n) at all! This is a significant improvement on the upper bound of Theorem 5, as that upper bound is proportional to t^2 .

By Theorem 8, there always is an $m < |\mathcal{X}|$ such that $\rho((I + \delta \underline{Q})^m) < 1$ for all $\delta \in \mathbb{R}_{>0}$ such that $\delta \|\underline{Q}\| < 2$. Thus, given such an m , we can easily improve Algorithm 1. After we have determined n and δ with Algorithm 1, we can simply determine the upper bound of Proposition 9. If $m(1 - \beta^k) < n(1 - \beta)$ (or $m < n(1 - \beta)$), then this upper bound is smaller than the desired maximal error ϵ , and we have found a tighter upper bound on the actual error. We can even go the extra mile and replace line 4 with a method that looks for the smallest possible $n \in \mathbb{N}$ that yields

$$m\delta^2 \|\underline{Q}\|^2 \|f\|_c (1 - \beta^k) \leq (1 - \beta)\epsilon,$$

where $k = \lceil n/m \rceil$ and $\delta = t/n$ —and therefore also β —are dependent of n . This method could yield a smaller n , but the time we gain by having to execute fewer iterations does not necessarily compensate the time lost by looking for a smaller n . In any case, to actually implement these improvements we need to be able to compute $\beta := \rho((I + \delta \underline{Q})^m)$.

Example 5. *For the simple case of Example 1, we can derive an analytical expression for $\rho((I + \delta \underline{Q}))$ that is valid for all $\delta \in \mathbb{R}_{\geq 0}$ such that $\delta \|\underline{Q}\| \leq 2$. Therefore, we can use Proposition 9 to a priori determine an upper bound for the error. If we choose $m = 1$, then $\epsilon_e = 0.767 \times 10^{-3}$ and $\epsilon_d = 1.79 \times 10^{-3}$. Note that $\epsilon_e < \epsilon$, so we can probably decrease the number of iterations n . As reported in Table 1, we find that $n = 6,133$ still suffices, and that this results in an approximation correct up to $\epsilon' = 0.560 \times 10^{-3}$, roughly two times smaller than the desired maximal error ϵ . The actual error is 0.0437×10^{-3} , roughly ten times smaller than ϵ .*

5.3 Approximating the coefficient of ergodicity

Unfortunately, determining the exact value of $\rho((I + \delta \underline{Q})^m)$ —and of $\rho(\underline{T})$ in general—turns out to be non-trivial and is often even impossible. Nevertheless, the following theorem gives some—actually computable—lower and upper bounds for the coefficient of ergodicity.

Theorem 10. *Let \underline{T} be a lower transition operator. Then*

$$\rho(\underline{T}) \leq \max \left\{ \max \{ [\overline{T}\mathbb{I}_A](x) - [\underline{T}\mathbb{I}_A](y) : x, y \in \mathcal{X} \} : \emptyset \neq A \subset \mathcal{X} \right\}, \quad (5)$$

$$\rho(\underline{T}) \geq \max \left\{ \max \{ [\underline{T}\mathbb{I}_A](x) - [\overline{T}\mathbb{I}_A](y) : x, y \in \mathcal{X} \} : \emptyset \neq A \subset \mathcal{X} \right\}. \quad (6)$$

The upper bound in Theorem 10 is particularly useful in combination with Proposition 9, as it allows us to replace $\beta := \rho((I + \delta \underline{Q})^m)$ with a guaranteed upper bound. Of course, this only makes sense if this upper bound is strictly smaller than one. The following proposition guarantees that, for ergodic lower transition rate operators \underline{Q} , this is always the case.

Proposition 11. *Let \underline{Q} be an ergodic lower transition rate operator. Then there is some $n < |\mathcal{X}|$ such that, for all $k \geq n$ and $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{>0}$ such that $\delta_i \|\underline{Q}\| < 2$ for all $i \in \{1, \dots, k\}$, the upper bound for $\rho(\Phi(\delta_1, \dots, \delta_k))$ that is given by Eqn. (5) is strictly smaller than one.*

5.4 Approximating limit values

The results that we have obtained earlier in this section naturally lead to a method to approximate $\underline{T}_\infty f := \lim_{t \rightarrow \infty} \underline{T}_t f$ up to some maximal error. This is an important problem in applications; for instance, Troffaes et al. (2015) try to determine $\underline{T}_\infty f$ for an ergodic lower transition rate operator that arises in their specific reliability analysis application. The method they use is rather ad hoc: they pick some t and n and then determine the uniform approximation $\Psi_t(n)f$. As $\|\Psi_t(n)f\|_v$ is small, they suspect that they are close to the actual limit value. They also observe that $\Psi_{2t}(4n)f$ only differs from $\Psi_t(n)f$ after the fourth significant digit, which they regard as further empirical evidence for the correctness of their approximation. While this ad hoc method seemingly works, the initial values for t and n have to be chosen somewhat arbitrarily. Also, this method provides no guarantee that the actual error is lower than some desired maximal error.

Theorem 8, Proposition 9, Theorem 10 and the following stopping criterion allow us to propose a method that corrects these two shortcomings.

Proposition 12. *Let \underline{Q} be an ergodic lower transition rate operator and let $f \in \mathcal{L}(\mathcal{X})$, $t \in \mathbb{R}_{\geq 0}$ and $\epsilon \in \mathbb{R}_{> 0}$. Let s denote a sequence $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^k \delta_i = t$ and, for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| \leq 2$. If $\|\underline{T}_t f - \Phi(s)f\| \leq \epsilon/2$ and $\|\Phi(s)f\|_c \leq \epsilon/2$, then for all $\Delta \in \mathbb{R}_{\geq 0}$:*

$$\left| \underline{T}_{t+\Delta} f - \frac{\max \Phi(s)f + \min \Phi(s)f}{2} \right| \leq \epsilon \quad \text{and} \quad \left| \underline{T}_\infty f - \frac{\max \Phi(s)f + \min \Phi(s)f}{2} \right| \leq \epsilon.$$

Without actually stating it, we mention that a similar—though less useful—stopping criterion can be proved for non-ergodic transition rate matrices as well.

Our method for determining $\underline{T}_\infty f$ is now relatively straightforward. Let \underline{Q} be an ergodic lower transition rate operator and fix some $f \in \mathcal{L}(\mathcal{X})$. We can then approximate $\underline{T}_\infty f$ up to any desired maximal error $\epsilon \in \mathbb{R}_{> 0}$ as follows. First, we look for some $m \in \mathbb{N}$ and some—preferably large— $\delta \in \mathbb{R}_{> 0}$ such that $\delta \|\underline{Q}\| < 2$ and

$$2m\delta^2 \|\underline{Q}\|^2 \|f\|_c \leq (1 - \beta)\epsilon,$$

where $\beta := \rho((I + \delta\underline{Q})^m)$. From Theorem 8, we know that a possible starting point for m is $|\mathcal{X}| - 1$. If we do not have an analytical expression for $\rho((I + \delta\underline{Q})^m)$, then we know from Proposition 11 that we can instead use the guaranteed upper bound of Theorem 10. If no such m and δ exist—for instance because the guaranteed upper bound on β is too conservative—then this method does not work. If on the other hand we do find such an m and δ , then we can keep on running the iterative step (line 7) of Algorithm 1 until we reach the first index $i \in \mathbb{N}$ such that $\|g_i\|_c \leq \epsilon/2$. By Propositions 9 and 12, we are now guaranteed that $(\max g_i + \min g_i)/2$ is an approximation of $\underline{T}_\infty f$ up to a maximal error ϵ .

Alternatively, we can fix a step size δ ourselves and use the method of Theorem 5 to compute ϵ' . In that case, we simply need to run the iterative scheme until we reach the first index i such that $\|g_i\|_c \leq \epsilon'$. By Proposition 12, we are then guaranteed that the error $(\max g_i + \min g_i)/2$ is an approximation of $\underline{T}_\infty f$ up to a maximal error $\epsilon = 2\epsilon'$. The same is true if we replace ϵ' by the error ϵ_e that is used in Proposition 9.

Example 6. *Using the analytical expressions of Example 2, we obtain $\underline{T}_\infty \mathbb{I}_1 \approx 9.5238095 \times 10^{-3}$.*

We want to approximate $\underline{T}_\infty \mathbb{I}_1$ up to a maximum error $\epsilon := 1 \times 10^{-6}$. We observe that $m = 1$ and $\delta \approx 3.485 \times 10^{-8}$ yield an ϵ_d that is lower than $\epsilon/2$. After 196,293,685 iterations, the norm of the

approximation is sufficiently small, resulting in the approximation $\underline{T}_\infty \mathbb{I}_1 = (9.524 \pm 0.001) \times 10^{-3}$. Alternatively, choosing $\delta = 1 \times 10^{-7}$ and continuing until $\|g_i\|_c \leq \epsilon'$ yields the approximation $\underline{T}_\infty \mathbb{I}_1 = (9.5242 \pm 0.0008) \times 10^{-3}$ after only 69,572,154 iterations.

Mimicking Troffaes et al. (2015), we also tried the heuristic method of increasing t and n until we observe empirical convergence. After some trying, we find that $t = 7$ and $n = 7 \cdot 250 = 1750$ already yield an approximation with sufficiently small error: $\|\underline{T}_\infty \mathbb{I}_1 - \Psi_7(1750) \mathbb{I}_1\| \approx 7 \times 10^{-7} < \epsilon$. Note however that for non-binary examples, where $\underline{T}_\infty f$ cannot be computed analytically, this heuristic approach is unable to provide a guaranteed bound.

6. Conclusion

We have improved an existing method and proposed a novel method to approximate $\underline{T}_t f$ up to any desired maximal error, where $\underline{T}_t f$ is the solution of the non-linear differential equation (1) that plays an essential role in the theory of imprecise continuous-time Markov chains. As guaranteed by our theoretical results, and as verified by our numerical examples, our methods outperform the existing method by Krak et al. (2016), especially if the lower transition rate operator is ergodic. For these ergodic lower transition rate operators, we also proposed a method to approximate $\lim_{t \rightarrow \infty} \underline{T}_t f$ up to any desired maximal error.

For the simple case of a binary state space, we observed in numerical examples that there is a rather large difference between the theoretically required number of iterations and the number of iterations that are empirically found to be sufficient. Similar differences can—although this falls beyond the scope of our present contribution—also be observed for the lower transition rate operator that is studied in (Troffaes et al., 2015). The underlying reason for these observed differences remains unclear so far. On the one hand, it could be that our methods are still on the conservative side, and that further improvements are possible. On the other hand, it might be that these differences are unavoidable, in the sense that guaranteed theoretical bounds come at the price of conservatism. We leave this as an interesting line of future research. Additionally, the performance of our proposed methods for systems with a larger state space deserves further inquiry.

Acknowledgments

Jasper De Bock is a Postdoctoral Fellow of the Research Foundation - Flanders (FWO) and wishes to acknowledge its financial support. The work in this paper was also partially supported by the H2020-MSCA-ITN-2016 UTOPIAE, grant agreement 722734. Finally, the authors would like to express their gratitude to three anonymous reviewers, for their time, effort and constructive feedback.

References

- W. J. Anderson. *Continuous-Time Markov Chains*. Springer-Verlag New York, 1991. doi:[10.1007/978-1-4612-3038-0](https://doi.org/10.1007/978-1-4612-3038-0).
- J. De Bock. The limit behaviour of imprecise continuous-time markov chains. *Journal of Nonlinear Science*, 27(1):159–196, 2017. doi:[10.1007/s00332-016-9328-3](https://doi.org/10.1007/s00332-016-9328-3).

- G. de Cooman, F. Hermans, and E. Quaeghebeur. Imprecise markov chains and their limit behavior. *Probability in the Engineering and Informational Sciences*, 23(4):597–635, 2009. doi:[10.1017/S0269964809990039](https://doi.org/10.1017/S0269964809990039).
- F. Hermans and G. de Cooman. Characterisation of ergodic upper transition operators. *International Journal of Approximate Reasoning*, 53(4):573 – 583, 2012. doi:[10.1016/j.ijar.2011.12.008](https://doi.org/10.1016/j.ijar.2011.12.008).
- T. Krak, J. De Bock, and A. Siebes. Imprecise continuous-time markov chains. 2016. arXiv Report 1611.05796 [math.PR].
- E. Seneta. *Non-negative Matrices and Markov Chains*. Springer-Verlag New York, 1981. doi:[10.1007/0-387-32792-4](https://doi.org/10.1007/0-387-32792-4).
- D. Škulj. Efficient computation of the bounds of continuous time imprecise markov chains. *Applied Mathematics and Computation*, 250:165–180, 2015. doi:[10.1016/j.amc.2014.10.092](https://doi.org/10.1016/j.amc.2014.10.092).
- M. C. M. Troffaes and G. de Cooman. *Lower Previsions*. Wiley, 2014.
- M. C. M. Troffaes, J. Gledhill, D. Skulj, and S. Blake. Using imprecise continuous time markov chains for assessing the reliability of power networks with common cause failure and non-immediate repair. In *Proceedings of the Ninth International Symposium on Imprecise Probability: Theories and Applications*, pages 287–294, 2015. URL <http://www.sipta.org/isipta15/data/paper/18.pdf>.
- D. Škulj and R. Hable. Coefficients of ergodicity for markov chains with uncertain parameters. *Metrika*, 76(1):107–133, 2013. doi:[10.1007/s00184-011-0378-0](https://doi.org/10.1007/s00184-011-0378-0).
- P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.

Appendix A. Extra material and proofs for Section 2

Definition 13. An operator $\|\cdot\|$ on a linear vector space \mathcal{L} is a norm if it maps \mathcal{L} to $\mathbb{R}_{\geq 0}$ and if for all $a, b \in \mathcal{L}$ and all $\mu \in \mathbb{R}$,

$$N1: \|\mu a\| = |\mu| \|a\|, \quad N3: \|a\| = 0 \Leftrightarrow a = 0.$$

$$N2: \|a + b\| \leq \|a\| + \|b\|,$$

If an operator only satisfies (N1) and (N2), then it is called a seminorm.

It can be immediately checked that the maximum norm $\|\cdot\|$ on $\mathcal{L}(\mathcal{X})$ is a proper norm, and similarly for the induced operator norm on non-negatively homogeneous operators from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$. For all $f \in \mathcal{L}(\mathcal{X})$ we define the variation seminorm $\|\cdot\|_v$ and the centred seminorm $\|\cdot\|_c$ as

$$\|f\|_v := \|f - \min f\| = \max\{|f(x) - \min f| : x \in \mathcal{X}\} = \max f - \min f \quad (7)$$

and

$$\|f\|_c := \|f - \tilde{f}\| = \max\{|f(x) - \tilde{f}| : x \in \mathcal{X}\} = (\max f - \min f)/2, \quad (8)$$

where $\tilde{f} := (\max f + \min f)/2$. Verifying that $\|\cdot\|_v$ and $\|\cdot\|_c$ are seminorms and not norms is straightforward.

Proposition 14. For all $f \in \mathcal{L}(\mathcal{X})$, all $\mu \in \mathbb{R}$ and any non-negatively homogeneous operator A ,

$$N4: \|f\|_c = \|f\|_v / 2,$$

$$N5: \|f\|_c \leq \|f\|,$$

$$N6: \|f + \mu\|_v = \|f\|_v,$$

$$N7: \|Af\| \leq \|A\| \|f\|,$$

$$N8: \|AB\| \leq \|A\| \|B\|.$$

Proof. Properties (N4), (N5) and (N6) follow almost immediately from the definitions of the centred and variation seminorms. Proofs for (N7) and (N8) can be found in (De Bock, 2017). ■

The following properties of lower transition operators will turn out to be useful in the proofs.

Proposition 15. Let \underline{T} , \underline{T}_1 , \underline{T}_2 , \underline{S}_1 and \underline{S}_2 be lower transition operators. Then for all $f, g \in \mathcal{L}(\mathcal{X})$ and all $\mu \in \mathbb{R}$:

$$L4: \min f \leq \underline{T}f \leq \bar{T}f \leq \max f;$$

$$L8: \|\underline{T}\| \leq 1;$$

$$L5: \underline{T}(f + \mu) = \underline{T}(f) + \mu;$$

$$L9: \|\underline{T}f - \underline{T}g\| \leq \|f - g\|;$$

$$L6: f \geq g \Rightarrow \underline{T}f \geq \underline{T}g \text{ and } \bar{T}f \geq \bar{T}g;$$

$$L10: \|\underline{T}A - \underline{T}B\| \leq \|A - B\|;$$

$$L7: |\underline{T}f - \underline{T}g| \leq \bar{T}(|f - g|);$$

$$L11: \|\underline{T}f\|_v \leq \|f\|_v;$$

L12: $\underline{T}_1 \underline{T}_2$ is a lower transition operator;

L13: $(\underline{T}_1 - \underline{T}_2)$ is a non-negatively homogeneous operator;

$$L14: \|\underline{T}_1 f - \underline{S}_1 f\|_c \leq \|\underline{T}_1 f - \underline{S}_1 f\| \leq \|\underline{T}_1 - \underline{S}_1\| \|f\|_c;$$

$$L15: \|\underline{T}_1 \underline{T}_2 f - \underline{S}_1 \underline{S}_2 f\|_c \leq \|\underline{T}_1 \underline{T}_2 f - \underline{S}_1 \underline{S}_2 f\| \leq \|\underline{T}_2 f - \underline{S}_2 f\| + \|\underline{T}_1 - \underline{S}_1\| \|\underline{S}_2 f\|_c.$$

Proof. Proofs for (L4)–(L10) and (L12) can be found in (De Bock, 2017).

(L11) follows almost immediately from (L4) and Eqn. (7):

$$\|\underline{T}f\|_v = \max \underline{T}f - \min \underline{T}f \leq \max f - \min f = \|f\|_v.$$

Note that for all $f \in \mathcal{L}(\mathcal{X})$ and all $\gamma \in \mathbb{R}_{\geq 0}$,

$$(\underline{T}_1 - \underline{T}_2)(\gamma f) = \underline{T}_1(\gamma f) - \underline{T}_2(\gamma f) = \gamma(\underline{T}_1 f - \underline{T}_2 f) = \gamma(\underline{T}_1 - \underline{T}_2)(f),$$

which proves (L13).

Next, we prove (L14). The first inequality follows from (N5). By (L13), $(\underline{T}_1 - \underline{S}_1)$ is a non-negatively homogeneous operator, such that

$$\begin{aligned} \|\underline{T}_1 f - \underline{S}_1 f\| &= \|\underline{T}_1 f - \tilde{f} - \underline{S}_1 f + \tilde{f}\| = \|\underline{T}_1(f - \tilde{f}) - \underline{S}_1(f - \tilde{f})\| \\ &= \|(\underline{T}_1 - \underline{S}_1)(f - \tilde{f})\| \leq \|\underline{T}_1 - \underline{S}_1\| \|f - \tilde{f}\| = \|\underline{T}_1 - \underline{S}_1\| \|f\|_c, \end{aligned}$$

where the second equality follows from (L5), the inequality follows from (L13) and (N7) and the last equality follows from Eqn. (8).

(L15) can be proved similarly. Again, the first inequality of (L15) follows from (N5). To prove the second inequality of (L15), we observe that

$$\begin{aligned} \|\underline{T}_1 \underline{T}_2 f - \underline{S}_1 \underline{S}_2 f\| &= \|\underline{T}_1 \underline{T}_2 f - \underline{T}_1 \underline{S}_2 f + \underline{T}_1 \underline{S}_2 f - \underline{S}_1 \underline{S}_2 f\| \\ &\leq \|\underline{T}_1 \underline{T}_2 f - \underline{T}_1 \underline{S}_2 f\| + \|\underline{T}_1 \underline{S}_2 f - \underline{S}_1 \underline{S}_2 f\| \\ &\leq \|\underline{T}_2 f - \underline{S}_2 f\| + \|\underline{T}_1 \underline{S}_2 f - \underline{S}_1 \underline{S}_2 f\| \\ &\leq \|\underline{T}_2 f - \underline{S}_2 f\| + \|\underline{T}_1 - \underline{S}_1\| \|\underline{S}_2 f\|_c, \end{aligned}$$

where the first inequality follows from (N2), the second inequality follows from (L9) and the third inequality follows from (L14). \blacksquare

A linear lower transition rate operator \underline{Q} —one for which (R2) holds with equality—can be identified with a matrix Q of dimension $|\mathcal{X}| \times |\mathcal{X}|$. This matrix is called a *transition rate matrix*, the (x, y) -component $Q(x, y)$ of which is equal to $[Q\mathbb{I}_y](x)$.

Lemma 16. *Let Q be a transition rate matrix. Then for all $x, y \in \mathcal{X}$ such that $x \neq y$,*

$$Q1: Q(x, y) \geq 0,$$

$$Q2: Q(x, x) = -\sum_{y \neq x} Q(x, y).$$

Also,

$$\|Q\| = 2 \max \{|Q(x, x)| : x \in \mathcal{X}\}.$$

Proof. Note that (Q1) follows immediately from (R4). From (R1), we find that for all $x \in \mathcal{X}$, $[Q\mathbb{I}_{\mathcal{X}}](x) = 0$. Using the linearity and (R1) yields

$$Q(x, x) = [Q\mathbb{I}_x](x) = \left[Q \left(1 - \sum_{y \neq x} \mathbb{I}_y \right) \right](x) = -\sum_{y \neq x} [Q\mathbb{I}_y](x) = \sum_{y \neq x} Q(x, y).$$

It is a matter of straightforward verification to prove that

$$\|Q\| = \max \left\{ \sum_{y \in \mathcal{X}} |Q(x, y)| : x \in \mathcal{X} \right\} = 2 \max \{|Q(x, x)| : x \in \mathcal{X}\}. \quad \blacksquare$$

Proposition 17 (Proposition 7.6 in (Krak et al., 2016)). *Let Q be a lower transition rate operator. The associated set of dominating rate matrices \underline{Q}_Q , defined as*

$$\underline{Q}_Q := \{Q \text{ a transition rate matrix} : (\forall f \in \mathcal{L}(\mathcal{X})) \underline{Q}f \leq Qf\},$$

is non-empty and bounded, and for all $f \in \mathcal{L}(\mathcal{X})$ there is some $Q \in \underline{Q}_Q$ such that $\underline{Q}f = Qf$.

Lemma 18 (Lemma G.3 in (Krak et al., 2016)). *Let \underline{Q} be a lower rate operator, then for any $Q \in \underline{Q}_Q$, $\|Q\| \leq \|\underline{Q}\|$.*

Proposition 19. *Let \underline{Q} be a lower transition rate operator. Then for all $f \in \mathcal{L}(\mathcal{X})$, all $\mu \in \mathbb{R}$ and all $x, y \in \mathcal{X}$ such that $x \neq y$:*

$$R5: \underline{Q}f \leq \overline{Q}f;$$

$$R8: 0 \leq \sum_{y \neq x} [\underline{Q}\mathbb{I}_x](y) \leq \|\underline{Q}\|/2;$$

$$R6: \underline{Q}(f + \mu) = \underline{Q}f;$$

$$R9: \|\underline{Q}\| = 2 \max\{|\underline{Q}\mathbb{I}_x](x)| : x \in \mathcal{X}\}.$$

$$R7: -\|\underline{Q}\|/2 \leq [\underline{Q}\mathbb{I}_x](x) \leq [\overline{Q}\mathbb{I}_x](x) \leq 0;$$

Proof. The properties (R5) and (R6) are proved in De Bock (2017). Hence, we only prove the remaining properties.

R7: By the conjugacy of \underline{Q} and \overline{Q} ,

$$\begin{aligned} [\overline{Q}\mathbb{I}_x](x) &= \left[\overline{Q} \left(1 - \sum_{z \neq x} \mathbb{I}_z \right) \right](x) = - \left[\underline{Q} \left(-1 + \sum_{z \neq x} \mathbb{I}_z \right) \right](x) \\ &\leq -[\underline{Q}(-1)](x) - \sum_{z \neq x} [\underline{Q}\mathbb{I}_z](x), \end{aligned}$$

where the inequality follows from (R2). By (R1) the first term is zero, such that

$$[\overline{Q}\mathbb{I}_x](x) \leq - \sum_{z \neq x} [\underline{Q}\mathbb{I}_z](x) \leq 0,$$

where the second inequality follows from (R4).

Recall that there is some $Q \in \mathcal{Q}_{\underline{Q}}$ such that $\underline{Q}\mathbb{I}_x = Q\mathbb{I}_x$. It holds that

$$[\underline{Q}\mathbb{I}_x](x) = [Q\mathbb{I}_x](x) = Q(x, x) \geq -\frac{\|Q\|}{2} \geq -\frac{\|\underline{Q}\|}{2},$$

where for the first inequality we used Lemma 16 and for the second inequality we used Lemma 18.

The property now follows by combining the obtained lower bound for $[\underline{Q}\mathbb{I}_x](x)$ and the obtained upper bound for $[\overline{Q}\mathbb{I}_x](x)$ with (R5).

R8: Recall from (R4) that $[Q\mathbb{I}_y](x)$ is non-negative if $y \neq x$, such that $\sum_{y \neq x} [\underline{Q}\mathbb{I}_y](x)$ is non-negative. Some manipulations yield

$$\begin{aligned} 0 \leq \sum_{y \neq x} [\underline{Q}\mathbb{I}_y](x) &\leq \left[\underline{Q} \left(\sum_{y \neq x} \mathbb{I}_y \right) \right](x) = - \left[\overline{Q} \left(- \sum_{y \neq x} \mathbb{I}_y \right) \right](x) \\ &= - \left[\overline{Q} \left(1 - \sum_{y \neq x} \mathbb{I}_y \right) \right](x) = -[\overline{Q}\mathbb{I}_x](x) \\ &\leq -[\underline{Q}\mathbb{I}_x](x), \end{aligned}$$

where the second inequality follows from (R2), the first equality follows from conjugacy, the second equality follows from (R6), and the final inequality follows from (R7). Also by (R7), we know that $-\underline{Q}\mathbb{I}_x](x)$ is non-negative and bounded above by $\|\underline{Q}\|/2$, hence

$$0 \leq \sum_{y \neq x} [\underline{Q}\mathbb{I}_y](x) \leq \frac{\|\underline{Q}\|}{2}.$$

R9: Let \underline{Q} be a lower transition rate operator. From (De Bock, 2017, R9) it follows that

$$\|\underline{Q}\| \leq 2 \max_{x \in \mathcal{X}} |[Q\mathbb{I}_x](x)|.$$

From (R7), however, we know that for all $x \in \mathcal{X}$, $|[Q\mathbb{I}_x](x)| \leq \|\underline{Q}\|/2$. Combining these two inequalities yields $\|\underline{Q}\| = 2 \max\{|[Q\mathbb{I}_x](x)| : x \in \mathcal{X}\}$. ■

Proof of Proposition 3. Fix some lower transition rate operator \underline{Q} and some $\delta \in \mathbb{R}_{\geq 0}$. We first prove that $\delta \|\underline{Q}\| \leq 2$ implies that the operator $(I + \delta \underline{Q})$ is a lower transition operator. The operator $(I + \delta \underline{Q})$ trivially satisfies (L2) and (L3), such that we only need to prove that it satisfies (L1). In order to do so, we fix some arbitrary $x \in \mathcal{X}$ and $f \in \mathcal{L}(\mathcal{X})$. It holds that

$$\begin{aligned} [(I + \delta \underline{Q})f](x) &= f(x) + \delta [Qf](x) \\ &= f(x) + \delta [Q(f - \min f)](x) \\ &= f(x) + \delta \left[\underline{Q} \left(\sum_{y \in \mathcal{X}} (f(y) - \min f) \mathbb{I}_y \right) \right](x) \\ &\geq f(x) + \delta (f(x) - \min f) [Q\mathbb{I}_x](x) + \delta \sum_{y \neq x} (f(y) - \min f) [Q\mathbb{I}_y](x) \\ &\geq f(x) + \delta (f(x) - \min f) [Q\mathbb{I}_x](x) \\ &\geq f(x) - \delta (f(x) - \min f) \frac{\|\underline{Q}\|}{2}, \end{aligned}$$

where the second equality follows (R6), the first inequality follows from (R2), the second inequality follows from (R4) and the third inequality follows from (R7). Recall that by assumption $\delta \|\underline{Q}\| \leq 2$, and therefore

$$[(I + \delta \underline{Q})f](x) \geq \min f.$$

Next, we prove the reverse implication. Assume that $(I + \delta \underline{Q})$ is a transition rate operator. By (R7) and (R9), there is some $x \in \mathcal{X}$ such that $[Q\mathbb{I}_x](x) = -\|\underline{Q}\|/2$. Hence,

$$[(I + \delta \underline{Q})\mathbb{I}_x](x) = \mathbb{I}_x(x) + \delta [Q\mathbb{I}_x](x) = 1 - \delta \frac{\|\underline{Q}\|}{2}.$$

If we now assume that $\delta \|\underline{Q}\| > 2$, then

$$[(I + \delta \underline{Q})\mathbb{I}_x](x) < 0 \leq \min \mathbb{I}_x,$$

which, by (L1), contradicts the initial assumption that $(I + \delta \underline{Q})$ is a lower transition operator. This allows us to conclude that if $(I + \delta \underline{Q})$ is a lower transition operator, then $\delta \|\underline{Q}\| \leq 2$. ■

Proof of Proposition 4. This proposition simply states (R9) of Proposition 19. ■

Proof of Example 1. We can immediately verify that \underline{Q} satisfies (R1)–(R4), such that it is indeed a lower transition rate operator. ■

Appendix B. Extra material for Section 3

We here give a slightly more detailed description of the differential equation of interest. Recall from the beginning of Section 3 that Škulj (2015) proved that for any lower transition rate operator \underline{Q} and any $f \in \mathcal{L}(\mathcal{X})$, the differential equation

$$\frac{d}{dt}f_t = \underline{Q}f_t$$

with initial condition $f_0 := f$ has a unique solution for all $t \in \mathbb{R}_{\geq 0}$. As mentioned by De Bock (2017), this differential equation actually determines a time-dependent operator \underline{T}_t : for all $t \in \mathbb{R}_{\geq 0}$, $\underline{T}_t f := f_t$. Even more, (De Bock, 2017, Proposition 9) states that for all $t \in \mathbb{R}_{\geq 0}$, the time-dependent operator \underline{T}_t itself satisfies the differential equation

$$\frac{d}{dt}\underline{T}_t = \underline{Q}\underline{T}_t \quad (9)$$

with initial condition $\underline{T}_0 := I$. De Bock (2017) also shows that this operator \underline{T}_t is a lower transition operator, and that it satisfies the semi-group property: for all $t_1, t_2 \in \mathbb{R}_{\geq 0}$,

$$\underline{T}_{t_1+t_2} = \underline{T}_{t_1}\underline{T}_{t_2}. \quad (10)$$

For a transition rate matrix, Eqn. (9) reduces to the linear differential equation

$$\frac{d}{dt}T_t = QT_t$$

with initial condition $T_0 := I$. This differential equation is essential to precise continuous-time Markov chains, and is often referred to as the *forward Kolmogorov* equation. The solution to this differential equation is called the *matrix exponential*, and is denoted by $T_t = e^{tQ}$.

Proof of Example 2. Fix any $\delta \in \mathbb{R}_{\geq 0}$ such that $\delta \|\underline{Q}\| \leq 2$, and let f be an arbitrary element of $\mathcal{L}(\mathcal{X})$. We immediately obtain that if $f(0) \geq f(1)$, then

$$\begin{aligned} [\Phi(\delta)f](0) &= f(0) - \delta \bar{q}_0(f(0) - f(1)) = f(0) - \delta \bar{q}_0 \|f\|_v, \\ [\Phi(\delta)f](1) &= f(1) + \delta \underline{q}_1(f(0) - f(1)) = f(1) + \delta \underline{q}_1 \|f\|_v. \end{aligned}$$

Similarly, if $f(0) \leq f(1)$, then

$$\begin{aligned} [\Phi(\delta)f](0) &= f(0) + \delta \underline{q}_0 \|f\|_v, \\ [\Phi(\delta)f](1) &= f(1) - \delta \bar{q}_1 \|f\|_v. \end{aligned}$$

Therefore, if $f(0) \geq f(1)$ then

$$[\Phi(\delta)f](0) - [\Phi(\delta)f](1) = \|f\|_v (1 - \delta(\bar{q}_0 + \underline{q}_1)),$$

and similarly if $f(0) \leq f(1)$, then

$$[\Phi(\delta)f](1) - [\Phi(\delta)f](0) = \|f\|_v (1 - \delta(\underline{q}_0 + \bar{q}_1)).$$

Consequently

$$f(0) \geq f(1) \Rightarrow \begin{cases} [\Phi(\delta)f](0) \geq [\Phi(\delta)f](1) & \text{if } \delta(\bar{q}_0 + \underline{q}_1) \leq 1, \\ [\Phi(\delta)f](0) \leq [\Phi(\delta)f](1) & \text{if } \delta(\bar{q}_0 + \underline{q}_1) \geq 1, \end{cases}$$

and

$$f(0) \leq f(1) \Rightarrow \begin{cases} [\Phi(\delta)f](0) \leq [\Phi(\delta)f](1) & \text{if } \delta(\underline{q}_0 + \bar{q}_1) \leq 1, \\ [\Phi(\delta)f](0) \geq [\Phi(\delta)f](1) & \text{if } \delta(\underline{q}_0 + \bar{q}_1) \geq 1. \end{cases}$$

Fix some $f \in \mathcal{L}(\mathcal{X})$, some $t \in \mathbb{R}_{\geq 0}$ and let $n \in \mathbb{N}$ such that

$$t(\bar{q}_0 + \underline{q}_1) \leq n, t(\underline{q}_0 + \bar{q}_1) \leq n \text{ and } t\|Q\| \leq 2n.$$

In this case, we can use the results obtained above to obtain an analytical expression for $\Psi_t(n)f$. If $f(0) \geq f(1)$, then

$$\begin{aligned} [\Psi_t(n)f](0) &= f(0) - \frac{t}{n}\bar{q}_0 \|f\|_v \sum_{i=0}^{n-1} \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)^i, \\ [\Psi_t(n)f](1) &= f(1) + \frac{t}{n}\underline{q}_1 \|f\|_v \sum_{i=0}^{n-1} \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)^i. \end{aligned}$$

Similarly, if $f(0) \leq f(1)$, then

$$\begin{aligned} [\Psi_t(n)f](0) &= f(0) + \frac{t}{n}\underline{q}_0 \|f\|_v \sum_{i=0}^{n-1} \left(1 - \frac{t}{n}(\underline{q}_0 + \bar{q}_1)\right)^i, \\ [\Psi_t(n)f](1) &= f(1) - \frac{t}{n}\bar{q}_1 \|f\|_v \sum_{i=0}^{n-1} \left(1 - \frac{t}{n}(\underline{q}_0 + \bar{q}_1)\right)^i. \end{aligned}$$

We now use Eqn. (2) to derive analytical expressions for the components of $\underline{T}_t f$. If $f(0) \geq f(1)$, then

$$\begin{aligned} [\underline{T}_t f](0) &= \lim_{n \rightarrow \infty} [\Psi_t(n)f](0) \\ &= \lim_{n \rightarrow \infty} \left(f(0) - \frac{t}{n}\bar{q}_0 \|f\|_v \sum_{i=0}^{n-1} \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)^i \right) \\ &= f(0) - \bar{q}_0 \|f\|_v \lim_{n \rightarrow \infty} \frac{t}{n} \sum_{i=0}^{n-1} \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)^i. \end{aligned}$$

Let us now assume that $\bar{q}_0 + \underline{q}_1 > 0$. If $t \neq 0$ and n is greater than the lower bounds mentioned above, the expression inside the parenthesis is bounded below by 0 and strictly bounded above by 1. Therefore,

$$\begin{aligned} [\underline{T}_t f](0) &= f(0) - \bar{q}_0 \|f\|_v \lim_{n \rightarrow \infty} \frac{t}{n} \frac{1 - \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)^n}{1 - \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)} \\ &= f(0) - \frac{\bar{q}_0}{\bar{q}_0 + \underline{q}_1} \|f\|_v \lim_{n \rightarrow \infty} \left(1 - \left(1 - \frac{t}{n}(\bar{q}_0 + \underline{q}_1)\right)^n\right) \\ &= f(0) - \frac{\bar{q}_0}{\bar{q}_0 + \underline{q}_1} \|f\|_v \left(1 - e^{-t(\bar{q}_0 + \underline{q}_1)}\right), \end{aligned}$$

and

$$[\underline{T}_t f](1) = f(1) + \frac{\underline{q}_1}{\bar{q}_0 + \underline{q}_1} \|f\|_v \left(1 - e^{-t(\bar{q}_0 + \underline{q}_1)}\right).$$

If $t = 0$, the obtained expressions hold trivially. Completely analogous, if $\underline{q}_0 + \bar{q}_1 > 0$, the case $f(0) \leq f(1)$ yields

$$\begin{aligned} [\underline{T}_t f](0) &= f(0) + \frac{\underline{q}_0}{\underline{q}_0 + \bar{q}_1} \|f\|_v \left(1 - e^{-t(\underline{q}_0 + \bar{q}_1)}\right) \\ [\underline{T}_t f](1) &= f(1) - \frac{\bar{q}_1}{\underline{q}_0 + \bar{q}_1} \|f\|_v \left(1 - e^{-t(\underline{q}_0 + \bar{q}_1)}\right). \end{aligned} \quad \blacksquare$$

Appendix C. Extra material and proofs for Section 4

In many of the following proofs, we frequently use the following lemma.

Lemma 20 (Lemma F.9 in (Krak et al., 2016)). *Let \underline{Q} be a lower transition rate operator. For any $\delta \in \mathbb{R}_{\geq 0}$, $\|\underline{T}_\delta - (I + \delta \underline{Q})\| \leq \delta^2 \|\underline{Q}\|^2$.*

Lemma 21. *Let \underline{Q} be a lower transition rate operator, $f \in \mathcal{L}(\mathcal{X})$ and $t \in \mathbb{R}_{\geq 0}$. Let $s := (\delta_1, \dots, \delta_k)$ be any sequence in $\mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^k \delta_i = t$ and, for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| \leq 2$. Then*

$$\|\underline{T}_t f - \Phi(s)f\| \leq \sum_{i=1}^k \delta_i^2 \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c$$

and

$$\|\underline{T}_t f - \Phi(s)f\| \leq \sum_{i=1}^k \delta_i^2 \|\underline{Q}\|^2 \left\| \underline{T}_{\Delta_{i-1}} f \right\|_c,$$

where $\Phi_0 := I$ and $\Delta_0 = 0$, and for all $i \in \{1, \dots, k\}$, $\Phi_i := (I + \delta_i \underline{Q})\Phi_{i-1}$ and $\Delta_i := \Delta_{i-1} + \delta_i$.

Proof. By the semi-group property of Eqn. (10),

$$\|\underline{T}_t f - \Phi(s)f\| = \|\underline{T}_{\delta_k} \underline{T}_{t-\delta_k} f - (I + \delta_k \underline{Q})\Phi_{k-1} f\|.$$

By Proposition 3, the operator $(I + \delta_i \underline{Q})$ is a lower transition operator for all $i \in \{1, \dots, k\}$. Even more, (L12) implies that the operator Φ_{i-1} is a lower transition operator for all $i \in \{1, \dots, k\}$. Recall that \underline{T}_{δ_k} and $\underline{T}_{t-\delta_k}$ are lower transition operators by definition, such that using (L15) and Lemma 20 yields

$$\begin{aligned} \|\underline{T}_t f - \Phi(s)f\| &\leq \|\underline{T}_{\delta_k} - (I + \delta_k \underline{Q})\| \|\Phi_{k-1} f\|_c + \|\underline{T}_{t-\delta_k} f - \Phi_{k-1} f\| \\ &\leq \delta_k^2 \|\underline{Q}\|^2 \|\Phi_{k-1} f\|_c + \|\underline{T}_{t-\delta_k} f - \Phi_{k-1} f\|. \end{aligned}$$

Repeated application of the same trick yields

$$\|\underline{T}_t f - \Phi(s)f\| \leq \sum_{i=1}^k \delta_i^2 \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c.$$

The second inequality of the statement can be proved in a completely similar manner. \blacksquare

Lemma 22. *Let \underline{Q} be a lower transition rate operator, $t \in \mathbb{R}_{\geq 0}$ and $f \in \mathcal{L}(\mathcal{X})$. If $\|f\|_c = 0$, $\|\underline{Q}\| = 0$ or $t = 0$, then $\|\underline{T}_t f - \Psi_t(0)f\| = \|\underline{T}_t f - f\| = 0$.*

Proof. If $\|f\|_c = 0$, then $\min f = \max f$, or equivalently f is a constant function. From (L4) it follows that in this case $\underline{T}_t f = f$ for all $t \in \mathbb{R}_{\geq 0}$. If $\|\underline{Q}\| = 0$, then $\underline{Q}g = 0$ for all $g \in \mathcal{L}(\mathcal{X})$. Therefore

$$\frac{d}{dt} \underline{T}_t f = \underline{Q} \underline{T}_t f = 0 \text{ for all } t \in \mathbb{R}_{\geq 0}.$$

Consequently, $\underline{T}_t f = \underline{T}_0 f = I f = f$. If $t = 0$, then we can simply use the initial condition: $\underline{T}_t f = \underline{T}_0 f = I f = f$.

In all three cases we find that $\underline{T}_t f = f$, and hence

$$\|\underline{T}_t f - \Psi_t(0)f\| = \|\underline{T}_t f - f\| = \|f - f\| = 0. \quad \blacksquare$$

Lemma 23. *Let \underline{Q} be a lower transition rate operator, $f \in \mathcal{L}(\mathcal{X})$, $t \in \mathbb{R}_{\geq 0}$, $\epsilon \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$, and define $\delta := t/n$. If*

$$n \geq \max \left\{ \frac{t \|\underline{Q}\|}{2}, \frac{t^2 \|\underline{Q}\|^2 \|f\|_c}{\epsilon} \right\},$$

then we are guaranteed that

$$\|\underline{T}_t f - \Psi_t(n)f\| \leq \epsilon' := \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{n-1} \|(I + \delta \underline{Q})^i f\|_c \leq \epsilon.$$

Proof. By Proposition 3, the operator $(I + \delta \underline{Q})$ is a lower transition operator if and only if $\delta \|\underline{Q}\| \leq 2$, or equivalently if and only if

$$n \geq \frac{t \|\underline{Q}\|}{2}. \quad (11)$$

From now on, we assume that n satisfies this inequality. Therefore, we may use Lemma 21 to yield

$$\|\underline{T}_t f - \Psi_t(n)f\| \leq \sum_{i=0}^{n-1} \delta^2 \|\underline{Q}\|^2 \|(I + \delta \underline{Q})^i f\|_c. \quad (12)$$

Note that for any $i \in \{0, \dots, n-1\}$, $(I + \delta \underline{Q})^i$ is a lower transition operator by (L12); hence it follows from (L11) that $\|(I + \delta \underline{Q})^i f\|_c \leq \|f\|_c$. Therefore

$$\|\underline{T}_t f - \Psi_t(n) f\| \leq \sum_{i=0}^{n-1} \delta^2 \|\underline{Q}\|^2 \|f\|_c = \frac{t^2 \|\underline{Q}\|^2 \|f\|_c}{n}.$$

It is now obvious that if

$$n \geq \frac{t^2 \|\underline{Q}\|^2 \|f\|_c}{\epsilon}, \quad (13)$$

then $\|\underline{T}_t f - \Psi_t(n) f\| \leq \epsilon$. It also follows almost immediately from Eqn. (12) that if n satisfies both Eqns. (11) and (13), then

$$\|\underline{T}_t f - \Psi_t(n) f\| \leq \epsilon' := \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{n-1} \|(I + \delta \underline{Q})^i f\|_c \leq \epsilon. \quad \blacksquare$$

Proof of Theorem 5. First, we assume $t = 0$, $\|\underline{Q}\| = 0$ or $\|f\|_c = 0$. In this case, $n = 0$ and $\delta = 0$. By Lemma 22, we find that

$$\|\underline{T}_t f - g_{(0)}\| = \|\underline{T}_t f - \Psi_t(0) f\| = 0 < \epsilon.$$

Next, we assume $t > 0$, $\|\underline{Q}\| > 0$ and $\|f\|_c > 0$. In this case, the integer n that is determined on line 4 of Algorithm 1 is just the lowest natural number that satisfies the requirement of Lemma 23, from which the stated follows immediately. \blacksquare

Lemma 24. Let \underline{Q} be a lower transition operator, $f \in \mathcal{L}(\mathcal{X})$, $t' \in \mathbb{R}_{\geq 0}$, $\epsilon \in \mathbb{R}_{> 0}$, $n, m, k \in \mathbb{N}$ and let $\delta_1, \dots, \delta_n$ be a sequence in $\mathbb{R}_{\geq 0}$. If (i) $k \leq m$, (ii) $k\delta_n + \sum_{i=1}^{n-1} m\delta_i = t'$, and (iii) for all $i \in \{1, \dots, n\}$, $\delta_i \|\underline{Q}\| \leq 2$ and

$$t' \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c \delta_i \leq \epsilon,$$

where $\Phi_0 := I$ and for all $i \in \{1, \dots, n-1\}$, $\Phi_i := (I + \delta_i \underline{Q})^m \Phi_{i-1}$; then

$$\begin{aligned} \|\underline{T}_{t'} f - \Phi_{m,k}(\delta_1, \dots, \delta_n) f\| &\leq \epsilon' := \sum_{i=1}^n \delta_i^2 \|\underline{Q}\|^2 \sum_{j=0}^{k_i-1} \|(I + \delta_i \underline{Q})^j \Phi_{i-1} f\|_c \\ &\leq \sum_{i=1}^n k_i \delta_i^2 \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c \leq \epsilon, \end{aligned}$$

where $k_i := m$ for all $i \in \{1, \dots, n-1\}$ and $k_n := k$.

Proof. Assume that (i) $1 \leq k \leq m$, (ii) $k\delta_n + \sum_{i=1}^{n-1} m\delta_i = t'$, and (iii) for all $i \in \{1, \dots, n\}$, $\delta_i \|\underline{Q}\| \leq 2$. Observe that by Proposition 3 and (L12), the operators $\Phi_0, \dots, \Phi_{n-1}$ are all lower transition operators. From Lemma 21, it follows that

$$\|\underline{T}_{t'} f - \Phi_{m,k}(\delta_1, \dots, \delta_n) f\| \leq \sum_{i=1}^n \delta_i^2 \|\underline{Q}\|^2 \sum_{j=0}^{k_i-1} \|(I + \delta_i \underline{Q})^j \Phi_{i-1} f\|_c. \quad (14)$$

Hence, it is obvious that the contribution of the i -th approximation step to (the upper bound of) the error is

$$\delta_i^2 \|\underline{Q}\|^2 \sum_{j=0}^{k_i-1} \|(I + \delta_i \underline{Q})^j \Phi_{i-1} f\|_c \leq k_i \delta_i^2 \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c, \quad (15)$$

where the inequality follows from (L11). We want that the contribution of the i -th approximation step to the error is proportional to its length $k_i \delta_i$. Therefore, we demand that the contribution of the i -th approximation step is bounded above by $k_i \delta_i \epsilon / t'$, which yields the condition

$$t' \delta_i \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c \leq \epsilon. \quad (16)$$

It is obvious that the conditions we have imposed on $\delta_1, \dots, \delta_n$ are those of the statement. Combining Eqns. (14), (15) and (16) yields

$$\begin{aligned} \|\underline{T}_t f - \Phi_{m,k}(\delta_1, \dots, \delta_n) f\| &\leq \epsilon' := \sum_{i=1}^n \delta_i^2 \|\underline{Q}\|^2 \sum_{j=0}^{k_i-1} \|(I + \delta_i \underline{Q})^j \Phi_{i-1} f\|_c \\ &\leq \sum_{i=1}^n k_i \delta_i^2 \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c \leq \epsilon. \end{aligned} \quad \blacksquare$$

Proof of Theorem 6. We use Algorithm 2 to determine n and k , and if applicable also k_i , δ_i and $g_{(i,j)}$. If $\|f\|_c = 0$, $\|\underline{Q}\| = 0$ or $t = 0$, then by Lemma 22

$$\|\underline{T}_t f - g_{(0,m)}\| = \|\underline{T}_t f - f\| = 0 < \epsilon.$$

We therefore assume that $\|f\|_c > 0$, $\|\underline{Q}\| > 0$ and $t > 0$, and let $\delta_1, \dots, \delta_n \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$ be determined by running Algorithm 2. Let $t' := k\delta_n + \sum_{i=1}^{n-1} m\delta_i \leq t$. It is then a matter of straightforward verification that $\delta_1, \dots, \delta_n$ and k satisfy the requirements of Lemma 24: (i) $1 \leq k \leq m$, (ii) $k\delta_n + \sum_{j=1}^{n-1} m\delta_j = t'$, and (iii) for all $i \in \{1, \dots, n\}$, $\delta_i \|\underline{Q}\| \leq 2$ and

$$t' \delta_i \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c \leq t \delta_i \|\underline{Q}\|^2 \|\Phi_{i-1} f\|_c = t \delta_i \|\underline{Q}\|^2 \|g_{(i-1,m)}\|_c \leq \epsilon.$$

Therefore,

$$\|\underline{T}_{t'} f - g_{(n,k)}\| \leq \sum_{i=1}^n \delta_i^2 \|\underline{Q}\|^2 \sum_{j=0}^{k_i-1} \|g_{(i,j)}\|_c \leq \sum_{i=1}^n k_i \delta_i^2 \|\underline{Q}\|^2 \|g_{(i-1,m)}\|_c \leq \epsilon. \quad (17)$$

If $t' = t$, this concludes the proof of the first part of the statement. If $t' < t$, we have that $\|g_{(n,k)}\|_c = 0$, which implies that there is some $\mu \in \mathbb{R}$ such that $g_{(n,k)} = \mu$. Hence, it follows that

$$\|\underline{T}_t f - g_{(n,k)}\| = \|\underline{T}_t f - \mu\| = \|\underline{T}_{t-t'} \underline{T}_{t'} f - \underline{T}_{t-t'} \mu\| \leq \|\underline{T}_{t'} f - \mu\| = \|\underline{T}_{t'} f - g_{(n,k)}\|,$$

where the second equality follows from Eqn. (10) and (L5) and where the inequality follows from (L9). Combined with Eqn. (17), this again implies the first part of the statement.

To prove the final part of the statement, we assume that $\|f\|_c > 0$, $\|\underline{Q}\| > 0$ and $t > 0$, and let $\delta_1, \dots, \delta_n \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$ be constructed by running Algorithm 2. We let n_u denote the number of iterations of the uniform method:

$$n_u := \left\lceil \max \left\{ \frac{t \|\underline{Q}\|}{2}, \frac{t^2 \|\underline{Q}\|^2 \|f\|_c}{\epsilon} \right\} \right\rceil.$$

If we let $\delta_u := t/n_u$, then obviously

$$0 < \delta_u \leq \min \left\{ t, \frac{2}{\|Q\|}, \frac{\epsilon}{t \|Q\|^2 \|f\|_c} \right\}.$$

We now consider two cases: $n = 1$ and $n > 1$. We start with the case $n = 1$. Let

$$\delta_1^* := \min \left\{ t, \frac{2}{\|Q\|}, \frac{\epsilon}{t \|Q\|^2 \|f\|_c} \right\}.$$

Since $n = 1$, it then holds that $t \leq m\delta_1^*$ and/or $\|g_{(1,m)}\|_c = 0$. We first assume that $t \leq m\delta_1^*$. Note that δ_1^* is strictly positive as we have assumed that $\|f\|_c$, $\|Q\|$ and t are strictly positive. We let $k := \lceil t/\delta_1^* \rceil$ and $\delta_1 := t/k$, such that

$$k = \left\lceil \frac{t}{\delta_1^*} \right\rceil = \left\lceil \max \left\{ 1, \frac{t\|Q\|}{2}, \frac{t^2\|Q\|^2\|f\|_c}{\epsilon} \right\} \right\rceil.$$

As in this case the definitions of n_u and k are equivalent, we find that $k + m(n - 1) = k = n_u$.

Next, we assume that $n = 1$ but $t > m\delta_1^*$. This can only be the case if $\|g_{(1,m)}\|_c = 0$ and

$$\delta_1 := \delta_1^* = \min \left\{ \frac{2}{\|Q\|}, \frac{\epsilon}{t \|Q\|^2 \|f\|_c} \right\}.$$

Therefore $\delta_u \leq \delta_1$, such that $n_u \geq t/\delta_1 > m$. As the total number of iterations is $k = m$, it immediately follows that $m(n - 1) + k = m < n_u$.

Next, we consider the case $n > 1$. For all $i \in \{1, \dots, n - 1\}$,

$$\delta_i := \min \left\{ \frac{2}{\|Q\|}, \frac{\epsilon}{t \|Q\|^2 \|\Phi_{i-1}f\|_c} \right\},$$

where $\Phi_0 := I$ and $\Phi_i := (I + \delta_i Q)^m \Phi_{i-1}$ and this definition is valid because we previously assumed that $\|f\|_c > 0$, $\|Q\| > 0$ and $t > 0$. Note that our definition of δ_i differs from that of line 6 in Algorithm 2: we have left out the upper bound $\Delta = t - \sum_{j=1}^{i-1} m\delta_j$ because this upper bound only plays a part for the final step δ_n . As by (L4) $\|\Phi_i f\|_c \leq \|\Phi_{i-1} f\|_c$, we find that

$$\delta_u \leq \delta_1 \leq \delta_2 \leq \dots \leq \delta_{n-1},$$

where the first inequality follows from the definition of δ_u . As the step sizes that are used are all larger than the uniform step size, we intuitively expect that the number of necessary iterations will be bounded above by n_u . To formally prove this, we again distinguish two sub-cases: $k\delta_n + \sum_{i=1}^{n-1} m\delta_i < t$ and $k\delta_n + \sum_{i=1}^{n-1} m\delta_i = t$.

We first consider the sub-case $k\delta_n + \sum_{i=1}^{n-1} m\delta_i < t$. This can only occur if $\|g_{(n,m)}\|_c = 0$ and $k = m$. As $m\delta_n < t - \sum_{i=1}^{n-1} m\delta_i$ and $\|g_{(n-1,m)}\|_c = \|\Phi_{n-1}f\|_c > 0$,

$$\delta_n = \left\{ \frac{2}{\|Q\|}, \frac{\epsilon}{t \|Q\|^2 \|\Phi_{n-1}f\|_c} \right\} \geq \delta_{n-1},$$

where the inequality follows from $\|\Phi_{n-2}f\|_c \geq \|\Phi_{n-1}f\|_c$. Note that

$$mn\delta_1 = (k + m(n-1))\delta_1 \leq k\delta_n + \sum_{i=1}^{n-1} m\delta_i < t = n_u\delta_u,$$

where the first inequality follows from the increasing character of $\delta_1, \dots, \delta_n$. If we divide both sides of the inequality by δ_1 , then we find that $mn < n_u\delta_u/\delta_1$. Using that $\delta_u \leq \delta_1$ now yields that the total number of iterations $k + (n-1)m = mn$ is strictly smaller than n_u .

Next, we consider the sub-case $k\delta_n + \sum_{i=1}^{n-1} m\delta_i = t$. Because $1 \leq k \leq m$ and $\delta_n > 0$, $\sum_{i=1}^{n-1} m\delta_i < t = n_u\delta_u$. Hence, there is some $n'_u < n_u$ such that $n'_u\delta_u < \sum_{i=1}^{n-1} m\delta_i \leq (n'_u + 1)\delta_u$. The final step size δ_n is derived from the remaining time

$$\begin{aligned} t - \sum_{i=1}^{n-1} m\delta_i &=: \Delta \geq n_u\delta_u - (n'_u + 1)\delta_u = (n_u - n'_u - 1)\delta_u, \\ \Delta &< n_u\delta_u - n'_u\delta_u = (n_u - n'_u)\delta_u, \end{aligned}$$

where the first inequality follows from $\sum_{i=1}^{n-1} m\delta_i \leq (n'_u + 1)\delta_u$ and the second inequality follows from $\sum_{i=1}^{n-1} m\delta_i > n'_u\delta_u$. We first determine the maximal allowable final step size

$$\delta_n^* := \min \left\{ \Delta, \frac{2}{\|Q\|}, \frac{\epsilon}{t \|Q\|^2 \|\Phi_{n-1}f\|_c} \right\},$$

and then determine the actual final step size as $\delta_n := \Delta/k$, with $1 \leq k := \lceil \Delta/\delta_n^* \rceil \leq m$.

If $(n_u - n'_u - 1) > 0$, then $\Delta \geq (n_u - n'_u - 1)\delta_u \geq \delta_u$. Therefore, and because the two other upper bounds of δ_n^* are also greater than δ_u , we find that $\delta_n^* \geq \delta_u$. From this, we infer that $k = \lceil \Delta/\delta_n^* \rceil \leq \lceil \Delta/\delta_u \rceil$. As $\Delta < (n_u - n'_u)\delta_u$, we now find that $k \leq (n_u - n'_u)$. Note that

$$m(n-1)\delta_1 \leq \sum_{i=1}^{n-1} m\delta_i \leq (n'_u + 1)\delta_u,$$

where the first inequality follows from the non-decreasing character of $\delta_1, \dots, \delta_{n-1}$. Dividing both sides of the inequality by δ_1 and using $\delta_u \leq \delta_1$ yields $m(n-1) \leq n'_u + 1$.

If $m(n-1) < n'_u + 1$, then combining this strict inequality with the obtained upper bound for k yields

$$k + m(n-1) < (n_u - n'_u) + (n'_u + 1) = n_u + 1,$$

which implies that $k + m(n-1) \leq n_u$, as desired.

If $m(n-1) = n'_u + 1$, then

$$\Delta = t - \sum_{i=1}^{n-1} m\delta_i \leq t - \sum_{i=1}^{n-1} m\delta_u = (n_u - n'_u - 1)\delta_u,$$

where the inequality is allowed because $\delta_u \leq \delta_1, \dots, \delta_{n-1}$. As we previously proved that $\Delta \geq (n_u - n'_u - 1)\delta_u$, we obtain that $m(n-1) = n'_u + 1$ implies that $\Delta = (n_u - n'_u - 1)\delta_u$. As $\delta_n^* \geq \delta_u$, in this case we are guaranteed that $k = \lceil \Delta/\delta_n^* \rceil = \lceil (n_u - n'_u - 1)\delta_u/\delta_n^* \rceil \leq (n_u - n'_u - 1)$. Hence, we again find that

$$k + m(n-1) \leq (n_u - n'_u - 1) + (n'_u + 1) = n_u,$$

as desired.

If $(n_u - n'_u - 1) = 0$, then $\Delta < (n_u - n'_u)\delta_u = \delta_u$. As the two other upper bounds on δ_n^* are greater than δ_u , this implies that $\delta_n^* = \Delta$. Consequently, $k = \lceil \Delta/\delta_n^* \rceil = \lceil \Delta/\Delta \rceil = 1$. Note that

$$m(n-1)\delta_1 \leq \sum_{i=1}^{n-1} m\delta_i < n_u\delta_u,$$

from which it follows that $m(n-1) < n_u$. Hence, we find that $k + m(n-1) < 1 + n_u$, and therefore also, once more, that $k + m(n-1) \leq n_u$. This concludes the proof. \blacksquare

Appendix D. A more thorough look at ergodicity

Before we prove the results of Section 5, we need to properly introduce the ergodicity of lower transition (rate) operators. We explicitly chose not to do this in the main text, as the main focus of this contribution is approximating $\underline{T}_t f$. Nevertheless, we now give a brief overview of the relevant literature, limiting ourselves to the qualitative point of view of [de Cooman et al. \(2009\)](#), [Hermans and de Cooman \(2012\)](#) and [De Bock \(2017\)](#).

D.1 Qualitatively characterising ergodicity of lower transition operators

Recall that a lower transition rate operator is ergodic if and only if $\underline{T}_t f$ converges to a constant function for all $f \in \mathcal{L}(\mathcal{X})$. [Hermans and de Cooman \(2012\)](#) say something similar for lower transition operators.

Definition 25. A lower transition operator \underline{T} is ergodic if, for all $f \in \mathcal{L}(\mathcal{X})$, the limit $\lim_{n \rightarrow \infty} \underline{T}^n f$ exists and is a constant function.

The condition of this definition can, in general, not be checked in practice. Nonetheless, [Hermans and de Cooman \(2012\)](#) provide a necessary and sufficient condition for the ergodicity of a lower transition operator, based on the following definition.

Definition 26. The lower transition operator \underline{T} is regularly absorbing if it is (i) top class regular, i.e.

$$\mathcal{X}_{PA} := \{x \in \mathcal{X} : (\exists n \in \mathbb{N})(\forall y \in \mathcal{X}) [\overline{T}^n \mathbb{I}_x](y) > 0\} \neq \emptyset,$$

and (ii) top class absorbing, i.e.

$$(\forall y \in \mathcal{X} \setminus \mathcal{X}_{PA})(\exists n \in \mathbb{N}) [\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}}](y) > 0.$$

Proposition 27 (Proposition 3 from [Hermans and de Cooman, 2012](#)). The lower transition operator \underline{T} is ergodic if and only if it is regularly absorbing.

[de Cooman et al. \(2009\)](#) mention an equivalent way of looking at top class regularity that uses the ternary accessibility relation $\cdot \rightsquigarrow \cdot$.

Definition 28. Let \underline{T} be any lower transition operator. For all $x, y \in \mathcal{X}$ and all $n \in \mathbb{N}_0$, we say that x is possibly accessible from y in n steps, denoted by $y \overset{n}{\rightsquigarrow} x$, if and only if $[\overline{T}^n \mathbb{I}_x](y) > 0$. If there is some $n \in \mathbb{N}_0$ such that $y \overset{n}{\rightsquigarrow} x$, then the state x is simply said to be possibly accessible from the state y , denoted by $y \rightsquigarrow x$.

Lemma 29. *Let \underline{T} be a lower transition operator, $x, y \in \mathcal{X}$ and $n \in \mathbb{N}$. Then $y \overset{n}{\rightsquigarrow} x$ if and only if there is a sequence $y = x_0, \dots, x_n = x$ in \mathcal{X} such that for all $k \in \{1, \dots, n\}$, $[\overline{T}\mathbb{I}_{x_k}](x_{k-1}) > 0$.*

Proof. Follows immediately from (Hermans and de Cooman, 2012, Proposition 4). \blacksquare

It can be almost immediately verified—for instance using Lemma 29—that $\cdot \overset{\cdot}{\rightsquigarrow} \cdot$ satisfies the three defining properties of a ternary accessibility relation:

$$\text{A1: } (\forall x, y \in \mathcal{X}) \ x \overset{0}{\rightsquigarrow} y \Leftrightarrow x = y,$$

$$\text{A2: } (\forall x, y, z \in \mathcal{X})(\forall n, m \in \mathbb{N}_0) \ x \overset{n}{\rightsquigarrow} y \text{ and } y \overset{m}{\rightsquigarrow} z \Rightarrow x \overset{n+m}{\rightsquigarrow} z,$$

$$\text{A3: } (\forall x \in \mathcal{X})(\forall n \in \mathbb{N})(\exists y \in \mathcal{X}) x \overset{n}{\rightsquigarrow} y.$$

The following proposition is the reason why we introduced the accessibility relation $\cdot \overset{\cdot}{\rightsquigarrow} \cdot$.

Proposition 30 (Proposition 4.3 from (de Cooman et al., 2009)). *The lower transition operator \underline{T} is top class regular if and only if*

$$\mathcal{X}_{PA} = \{x \in \mathcal{X} : (\exists n \in \mathbb{N})(\forall k \geq n)(\forall y \in \mathcal{X}) y \overset{k}{\rightsquigarrow} x\} \neq \emptyset.$$

Lemma 31. *If the lower transition operator \underline{T} is top class regular, then for all $x \in \mathcal{X}_{PA}$, all $y \in \mathcal{X}_{PA}^c$ and all $k \in \mathbb{N}$,*

$$[\overline{T}^k \mathbb{I}_y](x) = 0 \quad \text{and} \quad [\underline{T}^k \mathbb{I}_{\mathcal{X}_{PA}}](x) = 1.$$

Proof. Let \underline{T} be a top class regular lower transition operator with regular top class \mathcal{X}_{PA} . We first prove the first equality. To this end, we fix some arbitrary $x \in \mathcal{X}_{PA}$ and $y \in \mathcal{X}_{PA}^c$. Assume ex-absurdo that there is some $k \in \mathbb{N}$ such that $[\overline{T}^k \mathbb{I}_y](x) > 0$. By Definition 28, this assumption is equivalent to $x \overset{k}{\rightsquigarrow} y$. By Proposition 30, there is some $n \in \mathbb{N}$ such that for all $n \leq \ell \in \mathbb{N}$ and $z \in \mathcal{X}$, $z \overset{\ell}{\rightsquigarrow} x$. As a consequence of (A2), we find that for all $z \in \mathcal{X}$, $z \overset{\ell+k}{\rightsquigarrow} y$, which in turn implies that $y \in \mathcal{X}_{PA}$. However, this obviously contradicts our initial assumption, such that for all $k \in \mathbb{N}$, $[\overline{T}^k \mathbb{I}_y](x) = 0$.

Next, we prove the second statement. From the conjugacy of \underline{T} and \overline{T} and (L5), it follows that

$$\underline{T}\mathbb{I}_{\mathcal{X}_{PA}} = -\overline{T}(-\mathbb{I}_{\mathcal{X}_{PA}}) = 1 - \overline{T}(1 - \mathbb{I}_{\mathcal{X}_{PA}}) = 1 - \overline{T}\mathbb{I}_{\mathcal{X}_{PA}^c}.$$

From the conjugacy of \underline{T} and \overline{T} and (L2), it follows that

$$\overline{T}\mathbb{I}_{\mathcal{X}_{PA}^c} \leq \sum_{z \in \mathcal{X}_{PA}^c} \overline{T}\mathbb{I}_z.$$

From the—already proven—first equality of the statement, we know that $\sum_{z \in \mathcal{X}_{PA}^c} [\overline{T}\mathbb{I}_z](x) = 0$, hence

$$[\underline{T}\mathbb{I}_{\mathcal{X}_{PA}}](x) = 1 - [\overline{T}\mathbb{I}_{\mathcal{X}_{PA}^c}](x) \geq 1 - \sum_{z \in \mathcal{X}_{PA}^c} [\overline{T}\mathbb{I}_z](x) = 1.$$

Note that by (L4), $[\underline{T}\mathbb{I}_{\mathcal{X}_{PA}}](x) \leq \max \mathbb{I}_{\mathcal{X}_{PA}} = 1$. By combining the two obtained inequalities, we find that the second equality of the statement holds for $k = 1$: $[\underline{T}\mathbb{I}_{\mathcal{X}_{PA}}](x) = 1$. Next, fix some $k > 1$, and assume that the second equality holds for all $1 \leq \ell \leq k - 1$. Then by the induction

hypothesis and (L4), $\mathbb{I}_{\mathcal{X}_{PA}} \leq \underline{T}^{k-1} \mathbb{I}_{\mathcal{X}_{PA}}$. By (L6), this implies that $\underline{T} \mathbb{I}_{\mathcal{X}_{PA}} \leq \underline{T}^k \mathbb{I}_{\mathcal{X}_{PA}}$. As by the induction hypothesis $[\underline{T} \mathbb{I}_{\mathcal{X}_{PA}}](x) = 1$, we find that $[\underline{T}^k \mathbb{I}_{\mathcal{X}_{PA}}](x) \geq 1$. It immediately follows from (L4) and (L12) that $\underline{T}^k \mathbb{I}_{\mathcal{X}_{PA}} \leq 1$. Hence, we have shown that $[\underline{T}^k \mathbb{I}_{\mathcal{X}_{PA}}](x) = 1$, which finalises the proof. \blacksquare

The following proposition is an altered statement of (Hermans and de Cooman, 2012, Proposition 6).

Proposition 32. *Let \underline{T} be a top class regular lower transition operator. Then \underline{T} is top class absorbing if and only if $B_n = \mathcal{X}$, where $\{B_k\}_{k \in \mathbb{N}_0}$ is the sequence defined by the initial condition $B_0 := \mathcal{X}_{PA}$ and, for all $k \in \mathbb{N}_0$, by the recursive relation*

$$B_{k+1} := B_k \cup \{x \in \mathcal{X} \setminus B_k : [\underline{T} \mathbb{I}_{B_k}](x) > 0\} = \{x \in \mathcal{X} : [\underline{T} \mathbb{I}_{B_k}](x) > 0\},$$

and where $n \leq |\mathcal{X} \setminus \mathcal{X}_{PA}|$ is the first index such that $B_n = B_{n+1}$.

Proof. Let \underline{T} be a top class regular lower transition operator with regular top class \mathcal{X}_{PA} . By (Hermans and de Cooman, 2012, Proposition 6), \underline{T} is top class absorbing if and only if $A_n = \emptyset$, where A_n is the set determined by the initial condition $A_0 := \mathcal{X} \setminus \mathcal{X}_{PA}$ and, for all $k \in \mathbb{N}_0$, by the recursive relation

$$A_{n+1} := \{x \in A_k : [\bar{T} \mathbb{I}_{A_k}](x) = 1\},$$

and where $n \leq |\mathcal{X} \setminus \mathcal{X}_{PA}|$ is the first index for which $A_n = A_{n+1}$. For any $k \in \mathbb{N}_0$,

$$\bar{T} \mathbb{I}_{A_k} = -\underline{T}(-\mathbb{I}_{A_k}) = 1 - \underline{T}(1 - \mathbb{I}_{A_k}) = 1 - \underline{T} \mathbb{I}_{\mathcal{X} \setminus A_k},$$

where the first equality follows from the conjugacy of \underline{T} and \bar{T} and the second equality follows from (L5). Therefore, for all $x \in A_k$, $[\bar{T} \mathbb{I}_{A_k}](x) = 1$ if and only if $[\underline{T} \mathbb{I}_{\mathcal{X} \setminus A_k}](x) = 0$. Observe that $A_{k+1} \subseteq A_k$ and define $B_k := \mathcal{X} \setminus A_k$ for all $k \in \mathbb{N}_0$. Note that for all $k \in \mathbb{N}_0$, $B_k \subseteq B_{k+1}$ and

$$B_{k+1} \setminus B_k = A_k \setminus A_{k+1} = \{x \in A_k : [\underline{T} \mathbb{I}_{\mathcal{X} \setminus A_k}](x) > 0\} = \{x \in \mathcal{X} \setminus B_k : [\underline{T} \mathbb{I}_{B_k}](x) > 0\}.$$

Observe that $B_0 = \mathcal{X} \setminus A_0 = \mathcal{X}_{PA}$ and by the previous equality, for all $k \in \mathbb{N}_0$,

$$B_{k+1} = B_k \cup \{x \in \mathcal{X} \setminus B_k : [\underline{T} \mathbb{I}_{B_k}](x) > 0\}.$$

We now prove by induction that

$$B_{k+1} = \{x \in \mathcal{X} : [\underline{T} \mathbb{I}_{B_k}](x) > 0\} \text{ for all } k \in \mathbb{N}_0.$$

First, we consider the case $k = 0$. Recall from Lemma 31 that $[\underline{T} \mathbb{I}_{\mathcal{X}_{PA}}](x_0) > 0$ for all $x_0 \in \mathcal{X}_{PA}$. Hence,

$$\begin{aligned} B_1 &= B_0 \cup \{x \in \mathcal{X} \setminus B_0 : [\underline{T} \mathbb{I}_{B_0}](x) > 0\} \\ &= \{x \in B_0 : [\underline{T} \mathbb{I}_{B_0}](x) > 0\} \cup \{x \in \mathcal{X} \setminus B_0 : [\underline{T} \mathbb{I}_{B_0}](x) > 0\} \\ &= \{x \in \mathcal{X} : [\underline{T} \mathbb{I}_{B_0}](x) > 0\}. \end{aligned}$$

Next, we fix some $i \in \mathbb{N}$ and assume that the equality holds for all $k < i$. We now prove that the equality then also holds for $k = i$. Observe that $B_{k-1} \subseteq B_k$ implies $\mathbb{I}_{B_{k-1}} \leq \mathbb{I}_{B_k}$, which by (L6)

implies that $\underline{T}\mathbb{I}_{B_{k-1}} \leq \underline{T}\mathbb{I}_{B_k}$. Therefore, for all $x \in B_k$, since the induction hypothesis implies that $[\underline{T}\mathbb{I}_{B_{k-1}}](x) > 0$, we find $[\underline{T}\mathbb{I}_{B_k}](x) \geq [\underline{T}\mathbb{I}_{B_{k-1}}](x) > 0$. Hence,

$$\begin{aligned} B_{k+1} &= B_k \cup \{x \in \mathcal{X} \setminus B_k : [\underline{T}\mathbb{I}_{B_k}](x) > 0\} \\ &= \{x \in B_k : [\underline{T}\mathbb{I}_{B_k}](x) > 0\} \cup \{x \in \mathcal{X} \setminus B_k : [\underline{T}\mathbb{I}_{B_k}](x) > 0\} \\ &= \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_{B_k}](x) > 0\}. \end{aligned} \quad \blacksquare$$

The observant reader might have noticed that our definitions of top class regularity and top class absorption differ slightly from those in [Hermans and de Cooman \(2012\)](#), but they are actually entirely equivalent. For top class regularity, we demand that there is some $n \in \mathbb{N}$ such that $\overline{T}^n \mathbb{I}_x > 0$. By [\(L4\)](#), for any $k \geq n$ it then holds that $\overline{T}^k \mathbb{I}_x > 0$, which is what [Hermans and de Cooman \(2012\)](#) demand. For top class absorption, [Hermans and de Cooman \(2012\)](#) demand that

$$(\forall y \in \mathcal{X}_{PA}^c)(\exists n \in \mathbb{N}) [\overline{T}^n \mathbb{I}_{\mathcal{X}_{PA}^c}](y) < 1,$$

where $\mathcal{X}_{PA}^c := \mathcal{X} \setminus \mathcal{X}_{PA}$. Note that $[\overline{T}^n \mathbb{I}_{\mathcal{X}_{PA}^c}](y) = 1 - [\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}}](y)$, such that their demand is equivalent to our demand

$$(\forall y \in \mathcal{X}_{PA}^c)(\exists n \in \mathbb{N}) [\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}}](y) > 0.$$

By Lemma [31](#), for all $n \in \mathbb{N}$ and all $y \in \mathcal{X}_{PA}$, $[\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}}](y) > 0$, such that we could actually demand that

$$(\forall y \in \mathcal{X})(\exists n \in \mathbb{N}) [\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}}](y) > 0.$$

D.2 Qualitatively characterising ergodicity of lower transition rate operators

We now turn to the ergodicity of imprecise continuous-time Markov chains. A first and thorough study of the quantitative aspects concerning ergodicity was conducted by [De Bock \(2017\)](#). We only recall the definitions and results from [\(De Bock, 2017\)](#) that will be relevant to us in the remainder.

Definition 33. A state $x \in \mathcal{X}$ is upper reachable from the state $y \in \mathcal{X}$, denoted by $y \xrightarrow{\overline{Q}} x$, if (i) $x = y$, or (ii) there is some sequence $y = x_0, \dots, x_n = x$ in \mathcal{X} of length $n + 1 \geq 2$ such that for all $k \in \{1, \dots, n\}$, $[\overline{Q}\mathbb{I}_{x_k}](x_{k-1}) > 0$.

Note that a state x is always upper reachable from itself! Rather remarkably, this definition of upper reachability is strikingly similar to the alternative condition of Lemma [29](#) for possible accessibility. The links between these two definition will be made more explicit later.

Lemma 34. Let \underline{Q} be a lower rate operator, and $x, y \in \mathcal{X}$ such that $x \neq y$. Then x is upper reachable from y if and only if there is some sequence $y = x_0, \dots, x_n = x$ in \mathcal{X} in which every state occurs at most once and for all $k \in \{1, \dots, n\}$, $[\underline{Q}\mathbb{I}_{x_k}](x_{k-1}) > 0$. Consequently, $n < |\mathcal{X}|$.

Proof. The forward implication follows almost immediately from Definition [33](#). Assume that $y \xrightarrow{\overline{Q}} x$, then by Definition [33](#) there is some sequence $y = x_0, \dots, x_n = x$ in \mathcal{X} such that for all $k \in \{1, \dots, n\}$, $[\overline{Q}\mathbb{I}_{x_k}](x_{k-1}) > 0$. Assume that there is a state $z \in \mathcal{X}$ that occurs more than once in this sequence. Then we can simply delete every element of the sequence from right after the the first occurrence of z up to and including the last occurrence of z , and still have a valid sequence. If we continue this way, then we end up with a sequence in which every state occurs at most once.

As every state occurs at most once, the length $n + 1$ of the sequence is lower than or equal to $|\mathcal{X}|$. Consequently, $n < |\mathcal{X}|$.

The reverse implication follows from the fact that the requirements of Definition 33 are trivially satisfied. \blacksquare

Lemma 35. *Let Q be a lower transition rate operator, and $x, y \in \mathcal{X}$ such that $y \xrightarrow{\overline{Q}} x$. Then there is an integer $n < |\mathcal{X}|$ such that for all $k \geq n$ and all $\delta_1, \dots, \delta_k \in \mathbb{R}_{>0}$ such that $\delta_i \|\underline{Q}\| < 2$ for all $i \in \{1, \dots, k\}$, there is a sequence $y = x_0, \dots, x_k = x$ in \mathcal{X} such that $[(I + \delta_i \overline{Q})\mathbb{I}_{x_i}](x_{i-1}) > 0$ for all $i \in \{1, \dots, k\}$.*

Proof. We first consider the special case $x = y$. For all $\delta \in \mathbb{R}_{>0}$ such that $\delta \|\underline{Q}\| < 2$,

$$[(I + \delta \overline{Q})\mathbb{I}_x](x) = \mathbb{I}_x(x) + \delta[\overline{Q}\mathbb{I}_x](x) = 1 + \delta[\overline{Q}\mathbb{I}_x](x) > 0,$$

where the inequality follows from (R7). Therefore, for all $k \in \mathbb{N}$ and all $\delta_1, \dots, \delta_k \in \mathbb{R}_{>0}$ such that for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| < 2$, we find that $[(I + \delta_i \overline{Q})\mathbb{I}_x](x) > 0$ for all $i \in \{1, \dots, k\}$.

Next, we consider the case $y \neq x$. From Lemma 34 we know that there is a sequence $S_y := (y = x_0, \dots, x_n = x)$ in \mathcal{X} such that every state occurs at most once—i.e. $n < |\mathcal{X}|$ —and for all $i \in \{1, \dots, n\}$, $[\overline{Q}\mathbb{I}_{x_i}](x_{i-1}) > 0$. We fix an arbitrary $k \geq n$ and an arbitrary sequence $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{>0}$ such that for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| < 2$. Note that for all $i \in \{1, \dots, n\}$,

$$0 < \delta_i[\overline{Q}\mathbb{I}_{x_i}](x_{i-1}) = \mathbb{I}_{x_i}(x_{i-1}) + \delta_i[\overline{Q}\mathbb{I}_{x_i}](x_{i-1}) = [(I + \delta_i \overline{Q})\mathbb{I}_{x_i}](x_{i-1}),$$

where the inequality follows from $0 < \delta_i$ and the first equality is true because—by construction— $x_i \neq x_{i-1}$. Also, from the previous we know that for all $i \in \{n + 1, \dots, k\}$, $[(I + \delta_i \overline{Q})\mathbb{I}_x](x) > 0$. Hence, appending the sequence S_y with $(k - n)$ times x yields a sequence $y = x_0, \dots, x_k = x$ in \mathcal{X} such that for all $i \in \{1, \dots, k\}$, $[(I + \delta_i \overline{Q})\mathbb{I}_{x_i}](x_{i-1}) > 0$. \blacksquare

Definition 36. A (non-empty) set of states $A \subseteq \mathcal{X}$ is lower reachable from the state x , denoted by $x \xrightarrow{Q} A$, if $x \in B_n$, where $\{B_k\}_{k \in \mathbb{N}_0}$ is the sequence that is defined by the initial condition $B_0 := A$ and for all $k \in \mathbb{N}_0$ by the recursive relation

$$B_{k+1} := B_k \cup \{y \in \mathcal{X} \setminus B_k : [\underline{Q}\mathbb{I}_{B_k}](y) > 0\},$$

and $n \leq |\mathcal{X} \setminus A|$ is the first index for which $B_k = B_{k+1}$.

Again, remark the striking similarity between Definition 36 and Proposition 32.

Definition 37. A lower transition rate operator \underline{Q} is regularly absorbing if it is (i) top class regular, i.e.

$$\mathcal{X}_R := \{x \in \mathcal{X} : (\forall y \in \mathcal{X}) y \xrightarrow{\overline{Q}} x\} \neq \emptyset,$$

and (ii) top class absorbing, i.e.

$$(\forall y \in \mathcal{X} \setminus \mathcal{X}_R) y \xrightarrow{\underline{Q}} \mathcal{X}_R.$$

Theorem 38 (Theorem 19 in (De Bock, 2017)). *A lower transition rate operator \underline{Q} is ergodic if and only if it is regularly absorbing.*

Not surprisingly, these necessary and sufficient conditions for the ergodicity of lower rate matrices are rather similar to the necessary and sufficient conditions for ergodicity of lower transition operators given in Proposition 27.

Appendix E. Extra material and proofs for Section 5

Before we give any proofs, we first define the coefficient of ergodicity of an upper transition operator \overline{T} :

$$\rho(\overline{T}) := \max\{\|\overline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\}. \quad (18)$$

Proposition 39. *Let \underline{T} and \underline{S} be lower transition operators. For any $f \in \mathcal{L}(\mathcal{X})$,*

$$C1: 0 \leq \rho(\underline{T}) \leq 1,$$

$$C3: \rho(\overline{T}) = \rho(\underline{T}),$$

$$C2: \|\underline{T}f\|_v \leq \rho(\underline{T}) \|f\|_v,$$

$$C4: \rho(\underline{T}\underline{S}) \leq \rho(\underline{T})\rho(\underline{S}),$$

Proof. C1: Follows immediately from (L4).

C2: If $\|f\|_v = 0$, then by (L4) $\|\underline{T}f\|_v = 0$, such that the stated holds. Therefore, we now assume—without loss of generality—that $\|f\|_v > 0$. Note that $0 \leq (f - \min f)/\|f\|_v \leq 1$. Combining this with—in that order—(N6), (L5), (L3), (N1) and Eqn. (4), we find that

$$\begin{aligned} \|\underline{T}f\|_v &= \|\underline{T}f - \min f\|_v = \|\underline{T}(f - \min f)\|_v = \left\| \|f\|_v \underline{T} \left(\frac{f - \min f}{\|f\|_v} \right) \right\|_v \\ &= \left\| \underline{T} \left(\frac{f - \min f}{\|f\|_v} \right) \right\|_v \|f\|_v \\ &\leq \rho(\underline{T}) \|f\|_v. \end{aligned}$$

C3: By Eqn. (4),

$$\begin{aligned} \rho(\underline{T}) &= \max\{\|\underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max\{\|1 - \underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max\{\|1 + \overline{T}(-f)\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max\{\|\overline{T}(1 - f)\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max\{\|\overline{T}g\|_v : g \in \mathcal{L}(\mathcal{X}), 0 \leq g \leq 1\} \\ &= \rho(\overline{T}), \end{aligned}$$

where the second equality follows from (N6), the third equality follows from the conjugacy of \underline{T} and \overline{T} , the fourth equality follows from (L5), the fifth equality follows from the fact that $0 \leq f \leq 1$ if and only if $0 \leq 1 - f \leq 1$, and the final equality follows from Eqn. (18).

C4: By Eqn. (4) and (C2),

$$\begin{aligned} \rho(\underline{T}\underline{S}) &= \max\{\|\underline{T}\underline{S}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &\leq \max\{\rho(\underline{T}) \|\underline{S}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \rho(\underline{T}) \max\{\|\underline{S}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} = \rho(\underline{T})\rho(\underline{S}). \quad \blacksquare \end{aligned}$$

Theorem 21 in Škulj and Hable (2013) highlights the usefulness of the coefficient of ergodicity.

Theorem 40 (Theorem 21 in (Škulj and Hable, 2013)). *A lower transition operator \underline{T} is ergodic if and only if there is some $k \in \mathbb{N}$ such that $\rho(\underline{T}^k) < 1$.*

Proposition 41. *Let \underline{T} be a lower transition operator. Then*

$$\rho(\underline{T}) = \max \{ \|\underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), \max f = 1, \min f = 0 \} \quad (19)$$

$$= \max \{ \|\underline{T}f\|_c : f \in \mathcal{L}(\mathcal{X}), -1 \leq f \leq 1 \} \quad (20)$$

$$= \max \{ \|\underline{T}f\|_c : f \in \mathcal{L}(\mathcal{X}), \max f = 1, \min f = -1 \}. \quad (21)$$

Proof of Proposition 41. Because of Eqn. (4), there is some $g \in \mathcal{L}(\mathcal{X})$ such that $0 \leq g \leq 1$ and $\|\underline{T}g\|_v = \rho(\underline{T})$. By (C2), $\|\underline{T}g\|_v \leq \rho(\underline{T}) \|g\|_v$, such that $\|g\|_v = 1$, or equivalently $\max g = 1$ and $\min g = 0$. Hence, it follows from Eqn. (4) that

$$\rho(\underline{T}) = \max \{ \|\underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), \max f = 1, \min f = 0 \}.$$

Next, manipulating Eqn. (4) yields

$$\begin{aligned} \rho(\underline{T}) &= \max \{ \|\underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \} \\ &= \max \left\{ \left\| \underline{T} \left(f - \frac{1}{2} \right) \right\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \right\} \\ &= \max \left\{ \frac{2}{2} \left\| \underline{T} \left(f - \frac{1}{2} \right) \right\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \right\} \\ &= \max \left\{ \frac{1}{2} \|\underline{T}(2f - 1)\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \right\} \\ &= \max \{ \|\underline{T}(2f - 1)\|_c : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \}, \end{aligned}$$

where the second equality follows from (N6) and (L5), the fourth equality follows from (N1) and (L3), and the final equality follows from Eqn. (8). Note that for all $f \in \mathcal{L}(\mathcal{X})$, $0 \leq f \leq 1$ is equivalent to $-1 \leq (2f - 1) \leq 1$. Hence,

$$\rho(\underline{T}) = \max \{ \|\underline{T}f\|_c : f \in \mathcal{L}(\mathcal{X}), -1 \leq f \leq 1 \}.$$

The proof of the final equality of the statement is now similar to that of the first. ■

The following lemma is a more general version of Lemma 29.

Lemma 42. *Let $k \in \mathbb{N}$ and $x, y \in \mathcal{X}$. For all arbitrary upper transition operators $\bar{T}_1, \dots, \bar{T}_k$, we define $\bar{T}_{1:k} := \bar{T}_k \cdots \bar{T}_1$. Then*

$$[\bar{T}_{1:k} \mathbb{I}_x](y) \geq [\bar{T}_1 \mathbb{I}_{z_1}](z_2) \cdots [\bar{T}_k \mathbb{I}_{z_k}](z_{k+1}),$$

for any sequence $y = z_{k+1}, \dots, z_1 = x$ in \mathcal{X} . Furthermore, $[\bar{T}_{1:k} \mathbb{I}_x](y) > 0$ if and only if there is some sequence $y = z_{k+1}, \dots, z_1 = x$ in \mathcal{X} such that for all $i \in \{1, \dots, k\}$, $[\bar{T}_i \mathbb{I}_{z_i}](z_{i+1}) > 0$.

Proof. This proof is a straightforward generalisation of the proof of Proposition 4 in (Hermans and de Cooman, 2012). Fix some $k \in \mathbb{N}$, some $x, y \in \mathcal{X}$ and some arbitrary upper transition operators $\bar{T}_1, \dots, \bar{T}_k$. We also define $\bar{T}_{1:k} := \bar{T}_k \cdots \bar{T}_1$, and note that by (L12) this is also an upper transition operator.

To prove the first part of the statement, we note that for all $i \in \{1, \dots, k\}$ and all $z_i, z_{i+1} \in \mathcal{X}$,

$$\bar{T}_i \mathbb{I}_{z_i} = \sum_{z \in \mathcal{X}} [\bar{T}_i \mathbb{I}_{z_i}](z) \mathbb{I}_z \geq [\bar{T}_i \mathbb{I}_{z_i}](z_{i+1}) \mathbb{I}_{z_{i+1}},$$

where the inequality is allowed because by (L4) the sum contains only non-negative terms. We fix any $z_2 \in \mathcal{X}$, and use (L6) and this inequality to yield

$$\bar{T}_{1:k}\mathbb{I}_x = \bar{T}_{1:k-1}\bar{T}_1\mathbb{I}_x \geq \bar{T}_{1:k-1}([\bar{T}_1\mathbb{I}_x](z_2)\mathbb{I}_{z_2}) = [\bar{T}_1\mathbb{I}_x](z_2)\bar{T}_{1:k-1}\mathbb{I}_{z_2},$$

where $\bar{T}_{1:k-1} := \bar{T}_k \cdots \bar{T}_2$ —which by (L12) is also an upper transition operator—and the final equality follows from (L3) and (L4). Repeated application of the same reasoning yields

$$[\bar{T}_{1:k}\mathbb{I}_x](y) \geq [\bar{T}_1\mathbb{I}_{z_1}](z_2) \cdots [\bar{T}_k\mathbb{I}_{z_k}](z_{k+1}),$$

where $z_{k+1} := y$, $z_1 := x$, and z_2, \dots, z_k are arbitrary elements of \mathcal{X} . This proves the first part of the statement.

The reverse implication of the second part of the statement follows immediately from the first part. We therefore only need to prove that the forward implication holds as well. To that end, we first note that

$$[\bar{T}_{1:k}\mathbb{I}_x](y) = \left[\bar{T}_k \left(\sum_{z_k \in \mathcal{X}} [\bar{T}_{2:k}\mathbb{I}_x](z_k)\mathbb{I}_{z_k} \right) \right](y) \leq \sum_{z_k \in \mathcal{X}} [\bar{T}_{2:k}\mathbb{I}_x](z_k)[\bar{T}_1\mathbb{I}_{z_k}](y),$$

where $\bar{T}_{2:k} := \bar{T}_k \cdots \bar{T}_2$ and the inequality follows from (L2). Repeating this same reasoning another $(k-2)$ times yields

$$[\bar{T}_{1:k}\mathbb{I}_x](y) \leq \sum_{z_2 \in \mathcal{X}} \sum_{z_3 \in \mathcal{X}} \cdots \sum_{z_k \in \mathcal{X}} [\bar{T}_1\mathbb{I}_x](z_2)[\bar{T}_2\mathbb{I}_{z_2}](z_3) \cdots [\bar{T}_k\mathbb{I}_{z_k}](y).$$

If now $[\bar{T}_{1:k}\mathbb{I}_x](y) > 0$, then—because all terms are non-negative due to (L4)—at least one of the terms of the sum on the right hand side has to be strictly positive. Therefore, $[\bar{T}_{1:k}\mathbb{I}_x](y) > 0$ implies that there is at least one sequence $y = z_{k+1}, \dots, z_1 = x$ in \mathcal{X} such that for all $i \in \{1, \dots, k\}$, $[\bar{T}_i\mathbb{I}_{z_i}](z_{i+1}) > 0$. ■

Lemma 43. *Let $k \in \mathbb{N}$ and $A \subseteq \mathcal{X}$. For all arbitrary lower transition operators $\underline{T}_1, \dots, \underline{T}_k$, we define $\underline{T}_{1:k} := \underline{T}_k \cdots \underline{T}_1$. Then*

$$c_1 \cdots c_k \mathbb{I}_{A_k} \leq \underline{T}_{1:k} \mathbb{I}_A \leq \mathbb{I}_{A_k}.$$

In this expression, $A_k \subseteq \mathcal{X}$ is derived from the initial condition $A_0 := A$ and, for all $i \in \{1, \dots, k\}$, from the recursive relation

$$A_i := \{x \in \mathcal{X} : [\underline{T}_i \mathbb{I}_{A_{i-1}}](x) > 0\}.$$

The non-negative real numbers c_1, \dots, c_k are defined as

$$c_i := \min \{[\underline{T}_i \mathbb{I}_{A_{i-1}}](x) : x \in A_i\} \text{ for all } i \in \{1, \dots, k\},$$

with the convention that the minimum of an empty set is zero. Also, $A_k = \emptyset$ if and only if $c_i = 0$ for some $i \in \{1, \dots, k\}$.

Proof. Let \underline{T} be an arbitrary lower transition operator, and fix an arbitrary $A \subset \mathcal{X}$. We define the set $A' := \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_A](x) > 0\}$. On the one hand, from (L4) it follows that $\underline{T}\mathbb{I}_A \leq \mathbb{I}_{A'}$. On the other hand, $\underline{T}\mathbb{I}_A \geq c\mathbb{I}_{A'}$, where we let

$$c := \min \{[\underline{T}\mathbb{I}_A](x) : x \in A'\},$$

with the convention that the minimum of an empty set is zero. Note that by (L4), $0 \leq c \leq 1$. Combining these two inequalities yields $c\mathbb{I}_{A'} \leq \underline{T}\mathbb{I}_A \leq \mathbb{I}_{A'}$. Proving the first part of the statement is now fairly trivial; we simply need to apply both inequalities and (L6) k times.

To prove the second part of the statement, we observe that $c_i = 0$ is equivalent to $A_i = \emptyset$. Therefore, we assume that there is some $i \in \{1, \dots, k\}$ for which $c_i = 0$ and $A_i = \emptyset$. If $c_k = 0$, then obviously $A_k = \emptyset$ and the stated is true. We therefore assume that $i < k$, and observe that by (L4), $\underline{T}_{i+1}\mathbb{I}_{A_i} = \underline{T}_{i+1}\mathbb{I}_\emptyset = 0$, and therefore $A_{i+1} = \emptyset$. Repeating the same reasoning, we find that $A_j = \emptyset$ and $c_j = 0$ for all $j \in \{i, \dots, k\}$, which proves the stated. ■

The following lemma is an alternate, slightly extended version of Proposition 32.

Lemma 44. *Let \underline{T} be a top class regular lower transition operator. Then \underline{T} is top class absorbing if and only if $B_n = \mathcal{X}$, where $\{B_i\}_{i \in \mathbb{N}_0}$ is the sequence defined by the initial condition $B_0 := \mathcal{X}_{PA}$ and the recursive relation*

$$B_i = B_{i-1} \cup \{x \in \mathcal{X} \setminus B_{i-1} : [\underline{T}\mathbb{I}_{B_{i-1}}](x) > 0\} \text{ for all } i \in \mathbb{N},$$

and where $n \leq |\mathcal{X} \setminus \mathcal{X}_{PA}|$ is the first index such that $B_n = B_{n+1}$. Alternatively, \underline{T} is top class absorbing if and only if there is some $m \in \mathbb{N}_0$ such that $\underline{T}^m \mathbb{I}_{\mathcal{X}_{PA}} > 0$, and in this case n is the lowest such m .

Proof. We first prove the forward implication. By Proposition 32, if \underline{T} is top class absorbing then $B_n = \mathcal{X}$, where the sequence $\{B_i\}_{i \in \mathbb{N}_0}$ is defined from the initial condition $B_0 := \mathcal{X}_{PA}$ and, for all $i \in \mathbb{N}$, from the recursive relation

$$B_i := B_{i-1} \cup \{x \in \mathcal{X} \setminus B_{i-1} : [\underline{T}\mathbb{I}_{B_{i-1}}](x) > 0\} = \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_{B_{i-1}}](x) > 0\},$$

and where $n \leq |\mathcal{X} \setminus \mathcal{X}_{PA}|$ is the first index such that $B_n = B_{n+1}$. We can immediately verify that $\mathcal{X}_{PA} = B_0 \subseteq B_1 \subseteq \dots \subseteq B_n = \mathcal{X}$ and $B_i \setminus B_{i-1} \neq \emptyset$ for all $i \in \{1, \dots, n\}$.

Observe that the sequence B_0, \dots, B_n satisfies the conditions of Lemma 43, such that or all $i \in \{1, \dots, n\}$,

$$c_1 \cdots c_i \mathbb{I}_{B_i} \leq \underline{T}^i \mathbb{I}_{\mathcal{X}_{PA}} \leq \mathbb{I}_{B_i},$$

where c_1, \dots, c_n are strictly positive real numbers because $\emptyset \neq B_1, \dots, B_n$. From this we infer that $\min \underline{T}^i \mathbb{I}_{\mathcal{X}_{PA}} > 0$ if and only if $B_i = \mathcal{X}$. As $B_n = \mathcal{X}$ and $B_0, \dots, B_{n-1} \neq B_n$, this confirms that indeed $\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}} > 0$ and that n is the lowest non-negative natural number for which this holds.

Next, we prove the reverse implication. Let B_0, \dots, B_n and n be defined as in the statement. From the definition, it is obvious that $B_{i-1} \subseteq B_i$ for all $i \in \mathbb{N}$. Also, if n is the first index such that $B_n = B_{n+1}$, then $B_{i-1} \neq B_i$ for all $i \in \{1, \dots, n\}$ and $B_n = B_{n+i}$ for all $i \in \mathbb{N}$. From $B_0 = \mathcal{X}_{PA}$ and $B_i \setminus B_{i-1} \neq \emptyset$ for all $i \in \{1, \dots, n\}$, we infer that indeed $n \leq |\mathcal{X} \setminus \mathcal{X}_{PA}|$. If $B_n = \mathcal{X}$, then the sequence B_0, \dots, B_n satisfies the conditions of Proposition 32, such that \underline{T} is indeed top class absorbing.

Let B_0, \dots, B_n, \dots be the sequence as defined in the statement. Similar to what we did in the proof of Proposition 32, we now verify using induction that

$$B_i = \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_{B_{i-1}}](x) > 0\} \text{ for all } i \in \mathbb{N}.$$

We first consider the case $i = 1$. By Lemma 31, we know that $[\underline{T}\mathbb{I}_{\mathcal{X}_{PA}}](x) > 0$ for all $x \in \mathcal{X}_{PA}$. Hence,

$$\begin{aligned} B_1 &= B_0 \cup \left\{x \in \mathcal{X} \setminus B_0 : [\underline{T}\mathbb{I}_{B_0}](x) > 0\right\} \\ &= \left\{x \in B_0 : [\underline{T}\mathbb{I}_{B_0}](x) > 0\right\} \cup \left\{x \in \mathcal{X} \setminus B_0 : [\underline{T}\mathbb{I}_{B_0}](x) > 0\right\} \\ &= \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_{B_0}](x) > 0\}, \end{aligned}$$

where the second equality follows from the initial condition $B_0 = \mathcal{X}_{PA}$. Fix some $k \in \{1, \dots, n-1\}$, and assume that the alternate definition holds for all $i \leq k$. We now argue that in that case the stated also holds for $i = k+1$. By the induction hypothesis, B_k contains all $x \in \mathcal{X}$ for which $[\underline{T}\mathbb{I}_{B_{k-1}}](x) > 0$. Also, it holds by definition that $B_{k-1} \subseteq B_k$. Using (L6), we infer from $\mathbb{I}_{B_k} \geq \mathbb{I}_{B_{k-1}}$ that $[\underline{T}\mathbb{I}_{B_k}](x) \geq [\underline{T}\mathbb{I}_{B_{k-1}}](x) > 0$ for all $x \in B_k$. Hence,

$$\begin{aligned} B_{k+1} &= B_k \cup \{x \in \mathcal{X} \setminus B_k : [\underline{T}\mathbb{I}_{B_k}](x) > 0\} \\ &= \{x \in B_k : [\underline{T}\mathbb{I}_{B_k}](x) > 0\} \cup \{x \in \mathcal{X} \setminus B_k : [\underline{T}\mathbb{I}_{B_k}](x) > 0\} \\ &= \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_{B_k}](x) > 0\}. \end{aligned}$$

Now that we know that

$$B_i = \{x \in \mathcal{X} : [\underline{T}\mathbb{I}_{B_{i-1}}](x) > 0\} \text{ for all } i \in \mathbb{N},$$

we observe that this equivalent definition of the sequence satisfies the conditions of the sequence in Lemma 43. Moreover, as $\emptyset \neq B_0 \subseteq B_1 \subseteq \dots$, it follows from the second part of Lemma 43 that $c_i > 0$ for all $i \in \mathbb{N}$. Also from Lemma 43, we know that

$$c_1 \cdots c_i \mathbb{I}_{B_i} \leq \underline{T}^i \mathbb{I}_{\mathcal{X}_{PA}} \leq \mathbb{I}_{B_i} \text{ for all } i \in \mathbb{N}.$$

Assume now that there is some $m \in \mathbb{N}_0$ such that $\underline{T}^m \mathbb{I}_{\mathcal{X}_{PA}} > 0$, and let n be the lowest such m . Then for all $y \in \mathcal{X} \setminus \mathcal{X}_{PA}$, $[\underline{T}^n \mathbb{I}_{\mathcal{X}_{PA}}](y) > 0$, such that the second condition of Definition 26 is satisfied and \underline{T} is indeed top class absorbing. If $n = 0$, then $\mathcal{X}_{PA} = \mathcal{X} = B_0$, and n is indeed the first index for which $B_n = B_{n+1}$. If $n > 0$, then from the strict positivity of c_1, \dots, c_n and the lower and upper bound for $\underline{T}^i \mathbb{I}_{\mathcal{X}_{PA}}$ we infer that $B_1, \dots, B_{n-1} \neq \mathcal{X}$ and $B_n = \mathcal{X}$. We deduce from the recursive relation between B_0, \dots, B_n, B_{n+1} that n is indeed the first index for which $B_n = B_{n+1}$, which finalises this proof. \blacksquare

Proof of Theorem 8. We first prove the forward implication. To this end, we let \underline{Q} be an ergodic lower transition rate operator, and $n := |\mathcal{X}| - 1$ —we ignore the case $|\mathcal{X}| = 1$, as this case is trivially ergodic. We furthermore fix some $k \geq n$ and some $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{>0}$ such that for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| < 2$. For all $i \in \{1, \dots, k\}$, we define $\underline{T}_i := (I + \delta_i \underline{Q})$. By Proposition 3, the operators $\underline{T}_1, \dots, \underline{T}_k$ are lower transition operators, such that by (L12) their composition $\underline{T}_{1:k} := \underline{T}_k \cdots \underline{T}_1$ is also a lower transition operator. Note that the same holds for their conjugate upper transition operators, defined as $\overline{T}_i := (I + \delta_i \overline{Q})$ and $\overline{T}_{1:k} := \overline{T}_k \cdots \overline{T}_1$.

We now assume ex-absurdo that $\rho(\Phi(\delta_1, \dots, \delta_k)) = \rho(\underline{T}_{1:k}) = 1$. As a consequence of Proposition 41, there is some $f^* \in \mathcal{L}(\mathcal{X})$ with $\min f^* = 0$ and $\max f^* = 1$ such that $\|\underline{T}_{1:k} f^*\|_v = 1$. By construction and (L4), there are now some $y_0, y_1 \in \mathcal{X}$ such that $[\underline{T}_{1:k} f^*](y_0) = 0$ and $[\underline{T}_{1:k} f^*](y_1) = 1$.

We define the—obviously non-empty—set

$$\mathcal{X}^* := \{x \in \mathcal{X} : f^*(x) = 0\},$$

and distinguish two cases: either $\mathcal{X}_R \cap \mathcal{X}^* \neq \emptyset$ or $\mathcal{X}_R \cap \mathcal{X}^* = \emptyset$.

We first consider the case $\mathcal{X}_R \cap \mathcal{X}^* \neq \emptyset$, and fix any arbitrary $x^* \in \mathcal{X}_R \cap \mathcal{X}^*$. Note that, by construction, $\mathbb{I}_{x^*} \leq 1 - f^*$. Using the conjugacy of $\underline{T}_{1:k}$ and $\overline{T}_{1:k}$ and (L6), we find that

$$\overline{T}_{1:k} \mathbb{I}_{x^*} \leq \overline{T}_{1:k} (1 - f^*) = 1 + \overline{T}_{1:k} (-f^*) = 1 - \underline{T}_{1:k} f^*,$$

where the first equality follows from (L5) and the second equality follows from the conjugacy. From the previous inequality and (L4), it follows that

$$0 \leq [\overline{T}_{1:k} \mathbb{I}_{x^*}](y_1) \leq 1 - [\underline{T}_{1:k} f^*](y_1) = 0,$$

and hence $[\overline{T}_{1:k} \mathbb{I}_{x^*}](y_1) = 0$. From Lemma 42, it now follows that that

$$0 = [\overline{T}_{1:k} \mathbb{I}_{x^*}](y_1) \geq \prod_{i=1}^k \overline{T}_i \mathbb{I}_{z_i}(z_{i+1}) = \prod_{i=1}^k [(I + \delta_i \overline{Q}) \mathbb{I}_{z_i}](z_{i+1}) \quad (22)$$

for any arbitrary sequence $y_1 = z_{k+1}, z_2, \dots, z_1 = x^*$ in \mathcal{X} . On the other hand, as $k \geq n = |\mathcal{X}| - 1$ and $x^* \in \mathcal{X}_R$ it follows from Lemma 35 that there exists a sequence $y_1 = x_{k+1}, x_k, \dots, x_1 = x^*$ in \mathcal{X} such that $[(I + \delta_i \overline{Q}) \mathbb{I}_{x_i}](x_{i+1}) > 0$ for all $i \in \{1, \dots, k\}$. This obviously contradicts Eqn. (22).

Next, we consider the case $\mathcal{X}_R \cap \mathcal{X}^* = \emptyset$. In this case, $c \mathbb{I}_{\mathcal{X}_R} \leq f^*$, where we let

$$c := \min\{f^*(x) : x \in \mathcal{X}_R\} > 0.$$

From Lemma 43, we know that

$$c_1 \cdots c_k \mathbb{I}_{A_k} \leq \underline{T}_{1:k} \mathbb{I}_{\mathcal{X}_R},$$

where $A_0 := \mathcal{X}_R$ and, for all $i \in \{1, \dots, k\}$,

$$A_i := \{x \in \mathcal{X} : [\underline{T}_i \mathbb{I}_{A_{i-1}}](x) > 0\} \quad \text{and} \quad c_i := \min\{[\underline{T}_i \mathbb{I}_{A_{i-1}}](x) : x \in A_i\}.$$

As $c > 0$ and $c \mathbb{I}_{\mathcal{X}_R} \leq f^*$, it follows from (L3) and (L6) that $c \underline{T}_{1:k} \mathbb{I}_{\mathcal{X}_R} \leq \underline{T}_{1:k} f^*$. Combining the two obtained inequalities yields

$$cc_1 \cdots c_k \mathbb{I}_{A_k}(y_0) \leq c[\underline{T}_{1:k} \mathbb{I}_{\mathcal{X}_R}](y_0) \leq [\underline{T}_{1:k} f^*](y_0) = 0.$$

From the second part of Lemma 43, it now follows that $y_0 \notin A_k$.

Nonetheless, we now prove that $A_k = \mathcal{X}$, an obvious contradiction. To that end, observe that for all $i \in \{1, \dots, k\}$,

$$\begin{aligned} A_i &= \{x \in \mathcal{X} : [(I + \delta_i \underline{Q}) \mathbb{I}_{A_{i-1}}](x) > 0\} \\ &= \{x \in A_{i-1} : [(I + \delta_i \underline{Q}) \mathbb{I}_{A_{i-1}}](x) > 0\} \cup \{x \in \mathcal{X} \setminus A_{i-1} : [(I + \delta_i \underline{Q}) \mathbb{I}_{A_{i-1}}](x) > 0\}. \end{aligned}$$

Note that for all $x_{i-1} \in A_{i-1}$, $\mathbb{I}_{A_{i-1}} \geq \mathbb{I}_{x_{i-1}}$. Also, from (R7) it follows that $[(I + \delta_i \underline{Q})\mathbb{I}_{x_{i-1}}](x_{i-1}) > 0$. Using (L6) allows us to conclude that for all $x_{i-1} \in A_{i-1}$, $[(I + \delta_i \underline{Q})\mathbb{I}_{A_{i-1}}](x_{i-1}) > 0$. Therefore,

$$\begin{aligned} A_i &= A_{i-1} \cup \{x \in \mathcal{X} \setminus A_{i-1} : [(I + \delta_i \underline{Q})\mathbb{I}_{A_{i-1}}](x) > 0\} \\ &= A_{i-1} \cup \{x \in \mathcal{X} \setminus A_{i-1} : \mathbb{I}_{A_{i-1}}(x) + \delta_i [\underline{Q}\mathbb{I}_{A_{i-1}}](x) > 0\} \\ &= A_{i-1} \cup \{x \in \mathcal{X} \setminus A_{i-1} : [\underline{Q}\mathbb{I}_{A_{i-1}}](x) > 0\}, \end{aligned}$$

where the third equality is allowed because $\delta_i > 0$. From this recursive relation, it is obvious that $\mathcal{X}_R \subseteq A_k$. Even more, we can prove that $\mathcal{X}_R^c \subseteq A_k$, which implies that $\mathcal{X}_R \cup \mathcal{X}_R^c = \mathcal{X} \subseteq A_k \subseteq \mathcal{X}$, and consequently $A_k = \mathcal{X}$. Indeed, note that the sequence A_0, \dots, A_k is equal to the first $(k+1)$ terms of the sequence $\{B_i\}_{i \in \mathbb{N}_0}$ that is defined in Definition 36 for $B_0 = \mathcal{X}_R$. As \underline{Q} was assumed to be ergodic and $k \geq |\mathcal{X}| - 1 \geq |\mathcal{X} \setminus \mathcal{X}_R|$, it follows from Definitions 36 and 37 and Theorem 38 that $\mathcal{X}_R^c \subseteq B_k$.

For both $\mathcal{X}_R \cap \mathcal{X}^* \neq \emptyset$ and $\mathcal{X}_R \cap \mathcal{X}^* = \emptyset$ we have obtained a contradiction, such that the ergodicity of \underline{Q} indeed implies the stated.

Next, we prove the reverse implication. Fix some lower transition rate operator \underline{Q} , and assume that there is some $k < |\mathcal{X}|$ and some $\delta_1, \dots, \delta_k \in \mathbb{R}_{>0}$ such that $\delta_i \|\underline{Q}\| < 2$ for all $i \in \{1, \dots, k\}$ and

$$\rho(\Phi(\delta_1, \dots, \delta_k)) < 1.$$

By Proposition 40 this implies that the lower transition operator $\underline{T}_{1:k} := (I + \delta_k \underline{Q}) \cdots (I + \delta_1 \underline{Q})$ is ergodic. By Proposition 27, the ergodicity of $\underline{T}_{1:k}$ is equivalent to $\underline{T}_{1:k}$ being regularly absorbing, in the sense that

- (i) $\mathcal{X}_{1:k} := \{x \in \mathcal{X} : (\exists n \in \mathbb{N})(\forall y \in \mathcal{X}) [(\underline{T}_{1:k})^n \mathbb{I}_x](y) > 0\} \neq \emptyset$;
- (ii) $(\forall y \in \mathcal{X} \setminus \mathcal{X}_{1:k})(\exists n \in \mathbb{N}) [(\underline{T}_{1:k})^n \mathbb{I}_{\mathcal{X}_{1:k}}](y) > 0$.

Fix some $x^* \in \mathcal{X}_{1:k}$, and let $m \in \mathbb{N}$ such that $(\underline{T}_{1:k})^m \mathbb{I}_{x^*} > 0$. Fix an arbitrary $y \in \mathcal{X}$. Then by Lemma 42, there exists a sequence $y = x_{m+1}, \dots, x_1 = x^*$ in \mathcal{X} such that for all $i \in \{1, \dots, m\}$, $[\underline{T}_{1:k} \mathbb{I}_{x_i}](x_{i+1}) > 0$. Again using Lemma 42, this implies that for all $i \in \{1, \dots, m\}$ there is a sequence $x_{i+1} = x_{i,k+1}, \dots, x_{i,1} = x_i$ in \mathcal{X} such that for all $j \in \{1, \dots, k\}$,

$$[(I + \delta_j \underline{Q})\mathbb{I}_{x_{i,j}}](x_{i,j+1}) > 0.$$

As such, we have now constructed one long sequence

$$y = x_{m,k+1}, x_{m,k}, \dots, x_{m,1} = x_{m-1,k+1}, x_{m-1,k}, \dots, x_{m-1,1} = x_{m-2,k+1}, \dots, x_{1,1} = x$$

in \mathcal{X} . From this sequence we remove all “loops” (as we previously did in the proof of Lemma 34), and denote this shortened sequence by $y = z_{n'+1}, \dots, z_1 = x^*$ with corresponding time steps $\delta'_{n'}, \dots, \delta'_1$. Then for all $i \in \{1, \dots, n'\}$,

$$0 < [(I + \delta'_i \underline{Q})\mathbb{I}_{z_i}](z_{i+1}) = \mathbb{I}_{z_i}(z_{i+1}) + \delta'_i [\underline{Q}\mathbb{I}_{z_i}](z_{i+1}) = \delta'_i [\underline{Q}\mathbb{I}_{z_i}](z_{i+1}).$$

As all δ'_i are strictly positive, we find that for all $i \in \{1, \dots, n'\}$, $[\underline{Q}\mathbb{I}_{z_i}](z_{i+1}) > 0$. By Definition 33, this means that $y \xrightarrow{\underline{Q}} x^*$. As y was an arbitrary element of \mathcal{X} and x^* an arbitrary element of $\mathcal{X}_{1:k}$, $\mathcal{X}_{1:k} \subseteq \mathcal{X}_R$ and hence \underline{Q} is top class regular. Furthermore, we can show that $\mathcal{X}_R \subseteq \mathcal{X}_{1:k}$, such

that $\mathcal{X}_R = \mathcal{X}_{1:k}$. To that end, assume that $\mathcal{X}_R \setminus \mathcal{X}_{1:k} \neq \emptyset$ and fix some arbitrary $x^* \in \mathcal{X}_R \setminus \mathcal{X}_{1:k}$. Then by Definition 37, $y \xrightarrow{Q} x^*$ for all $y \in \mathcal{X}$. By Lemmas 35 and 42, for all $y \in \mathcal{X}$ there is an integer n_y such that for all $\ell \geq n_y$, $[(\underline{T}_{1:k})^\ell \mathbb{I}_{x^*}](y) > 0$. Hence, if we let $m := \max\{n_y : y \in \mathcal{X}\}$, then $[(\underline{T}_{1:k})^m \mathbb{I}_{x^*}](y) > 0$ for all $y \in \mathcal{X}$. By Definition 26, this implies that $x^* \in \mathcal{X}_{1:k}$. However, this contradicts our assumption that $x^* \in \mathcal{X}_R \setminus \mathcal{X}_{1:k}$, such that $\mathcal{X}_R \setminus \mathcal{X}_{1:k} = \emptyset$ and hence indeed $\mathcal{X}_R \subseteq \mathcal{X}_{1:k}$.

We now show that (ii) implies that \underline{Q} is top class absorbing. Since $\underline{T}_{1:k}$ is top class regular and absorbing, and because $\mathcal{X}_{1:k} = \mathcal{X}_R$, it follows from Lemma 44 that there is some $m \in \mathbb{N}_0$ such that $(\underline{T}_{1:k})^m \mathbb{I}_{\mathcal{X}_R} > 0$. Also, we know that $B_m = \mathcal{X}$, where $B_0 = \mathcal{X}_R$ and

$$B_{i+1} := B_i \cup \{x \in \mathcal{X} \setminus B_i : [\underline{T}_{1:k} \mathbb{I}_{B_i}](x) > 0\} \text{ for all } i \in \{0, \dots, m-1\}.$$

For any $i \in \{0, \dots, m-1\}$ and any $x \in \mathcal{X}$, it follows from Lemma 43 that $[\underline{T}_{1:k} \mathbb{I}_{B_i}](x) > 0$ if and only if $x \in B_{i,k}$, where $B_{i,k}$ is derived from the initial condition $B_{i,0} := B_i$ and, for all $j \in \{1, \dots, k\}$, from the recursive relation

$$B_{i,j} = \{x \in \mathcal{X} : [(I + \delta_j \underline{Q}) \mathbb{I}_{B_{i,j-1}}](z) > 0\}.$$

Similar to what we did before, we can rewrite this recursive relation as

$$\begin{aligned} B_{i,j} &= \{x \in B_{i,j-1} : [(I + \delta_j \underline{Q}) \mathbb{I}_{B_{i,j-1}}](z) > 0\} \cup \{x \in \mathcal{X} \setminus B_{i,j-1} : [(I + \delta_j \underline{Q}) \mathbb{I}_{B_{i,j-1}}](z) > 0\} \\ &= \{x \in B_{i,j-1} : 1 + \delta_j [\underline{Q} \mathbb{I}_{B_{i,j-1}}](z) > 0\} \cup \{x \in \mathcal{X} \setminus B_{i,j-1} : \delta_j [\underline{Q} \mathbb{I}_{B_{i,j-1}}](z) > 0\}. \end{aligned}$$

As before, we can verify that $1 + \delta_j [\underline{Q} \mathbb{I}_{B_{i,j-1}}](z) > 0$ for all $x \in B_{i,j-1}$. Hence,

$$B_{i,j} = B_{i,j-1} \cup \{x \in \mathcal{X} \setminus B_{i,j-1} : [\underline{Q} \mathbb{I}_{B_{i,j-1}}](z) > 0\}.$$

This way, we have constructed a sequence of sets

$$B_0 = B_{0,0}, B_{0,1}, \dots, B_{0,k} = B_1 = B_{1,0}, B_{1,1}, \dots, B_{1,k} = B_2 = B_{2,0}, \dots, B_{m-1,k} = B_m$$

with $B_0 = \mathcal{X}_R$ and $B_m = \mathcal{X}$. Denote this sequence by A_0, \dots, A_{mk+1} and let $A_{mk+2} := \mathcal{X}$. Then $A_0 = \mathcal{X}_R$, $A_{mk+1} = A_{mk+2} = \mathcal{X}$ and for all $i \in \{0, \dots, mk+1\}$,

$$A_{i+1} = A_i \cup \{x \in \mathcal{X} \setminus A_i : [\underline{Q} \mathbb{I}_{A_i}](z) > 0\}.$$

Let $n \in \{0, \dots, mk+1\}$ be the first index for which $A_n = A_{n+1}$. From the recursive relation between $A_n, \dots, A_{mk+1}, A_{mk+2}$, we infer that $A_n = A_{n+1} = \dots = A_{mk+2} = \mathcal{X}$. Fix an arbitrary $y^* \in \mathcal{X} \setminus \mathcal{X}_R$. Then the sequence $\mathcal{X}_R = A_0, \dots, A_n, A_{n+1}$ satisfies the recursive relation of Definition 36 and $y^* \in \mathcal{X} = A_n$, so $y^* \xrightarrow{Q} \mathcal{X}_R$. As y^* was an arbitrary element of $\mathcal{X} \setminus \mathcal{X}_R$, it follows that \underline{Q} is top class absorbing.

We have proven that if there is some $k < |\mathcal{X}|$ and some sequence $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{>0}$ such that $\delta_i \|\underline{Q}\| < 2$ for all $i \in \{1, \dots, k\}$ and $\rho(\Phi(\delta_1, \dots, \delta_k)) < 1$, then \underline{Q} is both top class regular and top class absorbing. As an immediate consequence of Theorem 38, this implies that \underline{Q} is ergodic. ■

Proof of Proposition 9. From the requirements on δ , (L12) and Proposition 3, it follows that $(I + \delta \underline{Q})^i$ is a lower transition operator for all $i \in \mathbb{N}$. By Lemma 21,

$$\begin{aligned} \|\underline{T}_t f - \Psi_t(n)\| &\leq \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{n-1} \|(I + \delta \underline{Q})^i f\|_c \\ &= \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \|(I + \delta \underline{Q})^j (I + \delta \underline{Q})^{mi} f\|_c. \end{aligned}$$

We use (L11) to yield

$$\|\underline{T}_t f - \Psi_t(n)\| \leq m \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{k-1} \|(I + \delta \underline{Q})^{mi} f\|_c.$$

Next, we simply use (C2) and (C4) to yield

$$\|\underline{T}_t f - \Psi_t(n)\| \leq m \delta^2 \|\underline{Q}\|^2 \|f\|_c \sum_{i=0}^{k-1} \rho((I + \delta \underline{Q})^m)^i.$$

For any $a \in [0, 1)$ and any $\ell \in \mathbb{N}$, it is well known that

$$\sum_{i=0}^{\ell} a^i = \frac{1 - a^{\ell+1}}{1 - a} \leq \frac{1}{1 - a}.$$

If $\beta := \rho((I + \delta \underline{Q})^m) < 1$, then we can use this well-known relation to yield

$$\|\underline{T}_t f - \Psi_t(n)\| \leq m \delta^2 \|\underline{Q}\|^2 \|f\|_c \frac{1 - \beta^k}{1 - \beta} \leq \frac{m \delta^2 \|\underline{Q}\|^2 \|f\|_c}{1 - \beta}.$$

The proof for $\beta = \rho(\underline{T}_{m\delta})$ is entirely analogous. We can use the second inequality of Lemma 21, the semi-group property and (L11), which yields

$$\|\underline{T}_t f - \Psi_t(n)\| \leq m \delta^2 \|\underline{Q}\|^2 \sum_{i=0}^{k-1} \|(\underline{T}_{m\delta})^i f\|_c.$$

Next, we again use (C2) and (C4) to yield

$$\|\underline{T}_t f - \Psi_t(n)\| \leq m \delta^2 \|\underline{Q}\|^2 \|f\|_c \sum_{i=0}^{k-1} \rho(\underline{T}_{m\delta})^i. \quad \blacksquare$$

Proof of Example 5. Let $\delta \in \mathbb{R}_{\geq 0}$ such that $\delta \|\underline{Q}\| \leq 2$. Using Proposition 41 yields

$$\rho(\Phi(\delta)) = \max\{\|\Phi(\delta)f\|_v : f \in \mathcal{L}(\mathcal{X}), \max f = 1, \min f = 0\}.$$

In the special case of a binary state space, only two functions satisfy this requirement: \mathbb{I}_0 and \mathbb{I}_1 . Therefore

$$\rho(\Phi(\delta)) = \max\{ |[\Phi(\delta)\mathbb{I}_0](0) - [\Phi(\delta)\mathbb{I}_0](1)|, |[\Phi(\delta)\mathbb{I}_1](0) - [\Phi(\delta)\mathbb{I}_1](1)| \}.$$

Recall that in the Proof of Example 2 we proved that for all $\delta \in \mathbb{R}_{\geq 0}$ such that $\delta \|\underline{Q}\| \leq 2$ and all $f \in \mathcal{L}(\mathcal{X})$,

$$[\Phi(\delta)f](0) - [\Phi(\delta)f](1) = \begin{cases} \|f\|_v (1 - \delta(\bar{q}_0 + \underline{q}_1)) & \text{if } f(0) \geq f(1), \\ \|f\|_v (1 - \delta(\underline{q}_0 + \bar{q}_1)) & \text{if } f(0) \leq f(1). \end{cases}$$

As $\|\mathbb{I}_0\|_v = 1 = \|\mathbb{I}_1\|_v$, this yields

$$\rho(I + \delta \underline{Q}) = \rho(\Phi(\delta)) = \max \left\{ \left| 1 - \delta(\bar{q}_0 + \underline{q}_1) \right|, \left| 1 - \delta(\underline{q}_0 + \bar{q}_1) \right| \right\}. \quad \blacksquare$$

For the proof of Theorem 10, we need some definitions and results from the theory of imprecise probabilities. The reason for this is that, as de Cooman et al. (2009) already mention, the functional $[\underline{T}\cdot](x)$ is actually a coherent (conditional) lower expectation. For a more thorough discussion of coherent lower expectations—often also called coherent lower previsions—we refer to the seminal work of Walley (1991) and the more recent treatment of Troffaes and de Cooman (2014).

Definition 45. A functional \underline{E} that maps $\mathcal{L}(\mathcal{X})$ to \mathbb{R} is a coherent lower expectation if for all $f, g \in \mathcal{L}(\mathcal{X})$ and all $\mu \in \mathbb{R}_{\geq 0}$:

$$\underline{E1}: \underline{E}(f) \geq \min f;$$

$$\underline{E2}: \underline{E}(f + g) \geq \underline{E}(f) + \underline{E}(g);$$

$$\underline{E3}: \underline{E}(\mu f) = \mu \underline{E}(f).$$

The conjugate coherent upper expectation is defined for all $f \in \mathcal{L}(\mathcal{X})$ as

$$\bar{\underline{E}}(f) = -\underline{E}(-f).$$

If for all $f \in \mathcal{L}(\mathcal{X})$, $\bar{\underline{E}}(f) = \underline{E}(f) = E(f)$, then we call E a linear expectation. The reason for this terminology is that the inequality in (E2) can then be replaced by an equality, and the condition $\mu \in \mathbb{R}_{\geq 0}$ for (E3) can be relaxed to $\mu \in \mathbb{R}$.

The following corollary highlights the link between the components of a lower transition operator and coherent lower previsions.

Corollary 46. Let \underline{T} be a lower transition operator and $x \in \mathcal{X}$. Then the functional $[\underline{T}\cdot](x): f \in \mathcal{L}(\mathcal{X}) \mapsto [\underline{T}f](x)$ is a coherent lower prevision.

Proof. The operator $[\underline{T}\cdot](x)$ indeed maps $\mathcal{L}(\mathcal{X})$ to \mathbb{R} . Furthermore, (E1) follows from (L1), (E2) follows from (L2) and (E3) follows from (L3). Hence, the operator is indeed a coherent lower prevision. \blacksquare

For any coherent lower expectation \underline{E} , the set $\mathcal{M}(\underline{E})$ of dominating linear expectations, defined as

$$\mathcal{M}(\underline{E}) := \{E \text{ a linear expectation operator: } (\forall f \in \mathcal{L}(\mathcal{X})) \underline{E}(f) \leq E(f)\},$$

is non-empty. Moreover, from (Walley, 1991, Section 3.3.3) it follows that \underline{E} is the lower envelope of $\mathcal{M}(\underline{E})$, in the sense that for all $f \in \mathcal{L}(\mathcal{X})$,

$$\underline{E}(f) = \min\{E(f) : E \in \mathcal{M}(\underline{E})\}.$$

Lemma 47 (Alternative statement of Proposition 1 in Škulj and Hable (2013)). *If E_1 and E_2 are two linear expectation operators, then*

$$\max\{E_1(f) - E_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} = \max\{E_1(\mathbb{I}_A) - E_2(\mathbb{I}_A) : f \in \mathcal{L}(\mathcal{X}), \emptyset \neq A \subset \mathcal{X}\}.$$

Proof. Let E_1 and E_2 be any two linear expectation operators on $\mathcal{L}(\mathcal{X})$. Then

$$\begin{aligned} & \max\{E_1(f) - E_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max \left\{ \sum_{x \in \mathcal{X}} (E_1(\mathbb{I}_x) - E_2(\mathbb{I}_x)) f(x) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \right\} \\ &= \sum_{x \in A^*} (E_1(\mathbb{I}_x) - E_2(\mathbb{I}_x)) = E_1(\mathbb{I}_{A^*}) - E_2(\mathbb{I}_{A^*}), \end{aligned}$$

where $A^* \subset \mathcal{X}$ is defined as $A^* := \{x \in \mathcal{X} : E_1(\mathbb{I}_x) > E_2(\mathbb{I}_x)\}$. ■

Lemma 48. *If \underline{E}_1 and \underline{E}_2 are two coherent lower expectations on $\mathcal{L}(\mathcal{X})$, then*

$$\max\{\underline{E}_1(f) - \underline{E}_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \leq \max\{\bar{E}_1(\mathbb{I}_A) - \underline{E}_2(\mathbb{I}_A) : \emptyset \neq A \subset \mathcal{X}\}.$$

Proof. Define $\mathcal{M}_1 := \mathcal{M}(\underline{E}_1)$ and $\mathcal{M}_2 := \mathcal{M}(\underline{E}_2)$. Note that for all $f \in \mathcal{L}(\mathcal{X})$,

$$0 = \underline{E}_1(0) = \underline{E}_1(f - f) \geq \underline{E}_1(f) + \underline{E}_1(-f),$$

where the first equality follows from (E3)—with $\mu = 0$ and $f = 0$ —and the first inequality follows from (E2). Bringing the second term to the left hand side and using the conjugacy relation between \underline{E}_1 and \bar{E}_1 , we find $\bar{E}_1(f) \geq \underline{E}_1(f)$. Hence

$$\underline{E}_1(f) - \underline{E}_2(f) \leq \bar{E}_1(f) - \underline{E}_2(f),$$

and consequently

$$\max\{\underline{E}_1(f) - \underline{E}_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \leq \max\{\bar{E}_1(f) - \underline{E}_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\}.$$

Recall that for any $f \in \mathcal{L}(\mathcal{X})$, $\underline{E}_i(f) = \min_{E_i \in \mathcal{M}_i} E_i(f)$, so

$$\bar{E}_1(f) - \underline{E}_2(f) = \max_{E_1 \in \mathcal{M}_1} E_1(f) - \min_{E_2 \in \mathcal{M}_2} E_2(f) = \max_{E_1 \in \mathcal{M}_1} \max_{E_2 \in \mathcal{M}_2} E_1(f) - E_2(f).$$

We use the previous equality to rewrite the right hand side of the previous inequality:

$$\begin{aligned} & \max\{\bar{E}_1(f) - \underline{E}_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max \left\{ \max_{E_1 \in \mathcal{M}_1} \max_{E_2 \in \mathcal{M}_2} (E_1(f) - E_2(f)) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \right\} \\ &= \max_{E_1 \in \mathcal{M}_1} \max_{E_2 \in \mathcal{M}_2} \max \{ (E_1(f) - E_2(f)) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \}. \end{aligned}$$

Next, we use Lemma 47 to yield

$$\begin{aligned} & \max\{\underline{E}_1(f) - \underline{E}_2(f) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &\leq \max_{E_1 \in \mathcal{M}_1} \max_{E_2 \in \mathcal{M}_2} \max \{ (E_1(f) - E_2(f)) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1 \} \\ &= \max_{E_1 \in \mathcal{M}_1} \max_{E_2 \in \mathcal{M}_2} \max \{ (E_1(\mathbb{I}_A) - E_2(\mathbb{I}_A)) : \emptyset \neq A \subset \mathcal{X} \} \\ &= \max \left\{ \max_{E_1 \in \mathcal{M}_1} \max_{E_2 \in \mathcal{M}_2} (E_1(\mathbb{I}_A) - E_2(\mathbb{I}_A)) : \emptyset \neq A \subset \mathcal{X} \right\} \\ &= \max\{\bar{E}_1(\mathbb{I}_A) - \underline{E}_2(\mathbb{I}_A) : \emptyset \neq A \subset \mathcal{X}\}. \end{aligned}$$
■

Proof of Theorem 10. Fix some lower transition operator \underline{T} . The lower bound on $\rho(\underline{T})$ follows from the fact that for any $\emptyset \neq A \subset \mathcal{X}$, $0 \leq \mathbb{I}_A \leq 1$. Recall from Corollary 46 that for any $x \in \mathcal{X}$, $[\underline{T} \cdot](x)$ is a coherent lower prevision. Therefore, we can use Lemma 48 to yield the upper bound:

$$\begin{aligned} \rho(\underline{T}) &= \max\{\|\underline{T}f\|_v : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} \\ &= \max\left\{\max\{[\underline{T}f](x) - [\underline{T}f](y) : x, y \in \mathcal{X}\} : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\right\} \\ &= \max\left\{\max\{[\underline{T}f](x) - [\underline{T}f](y) : f \in \mathcal{L}(\mathcal{X}), 0 \leq f \leq 1\} : x, y \in \mathcal{X}\right\} \\ &\leq \max\left\{\max\{[\underline{T}\mathbb{I}_A](x) - [\underline{T}\mathbb{I}_A](y) : \emptyset \neq A \subseteq \mathcal{X}\} : x, y \in \mathcal{X}\right\} \\ &= \max\left\{\max\{[\underline{T}\mathbb{I}_A](x) - [\underline{T}\mathbb{I}_A](y) : x, y \in \mathcal{X}\} : \emptyset \neq A \subseteq \mathcal{X}\right\}. \end{aligned} \quad \blacksquare$$

Proof of Proposition 11. Define $\underline{T}_{1:k} := \Psi(\delta_1, \dots, \delta_k)$, then by Proposition 3 and ((L12)), $\underline{T}_{1:k}$ is a lower transition operator, so we let $\overline{T}_{1:k}$ be its conjugate upper transition operator. From Theorem 10, it follows that

$$\rho(\underline{T}_{1:k}) \leq \max\left\{\max\left\{[\overline{T}_{1:k}\mathbb{I}_A](x) - [\underline{T}_{1:k}\mathbb{I}_A](y) : x, y \in \mathcal{X}\right\} : \emptyset \neq A \subset \mathcal{X}\right\}.$$

Let $n := |\mathcal{X} \setminus \mathcal{X}_R|$ and assume ex-absurdo that there is some $k \geq n$ and some sequence $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{>0}$ such that $\delta_i \|\underline{Q}\| < 2$ for all $i \in \{1, \dots, k\}$, and

$$\max\left\{\max\left\{[\overline{T}_{1:k}\mathbb{I}_A](x) - [\underline{T}_{1:k}\mathbb{I}_A](y) : x, y \in \mathcal{X}\right\} : \emptyset \neq A \subset \mathcal{X}\right\} = 1.$$

This can only be true if there is some $A \subset \mathcal{X}$ such that $\max \overline{T}_{1:k}\mathbb{I}_A = 1$ and $\min \underline{T}_{1:k}\mathbb{I}_A = 0$. Consider any such A .

As \underline{Q} is ergodic, it follows from Definition 37 and Theorem 38 that $\mathcal{X}_R \neq \emptyset$. As in Definition 36, we define the sequence $\{B_i\}_{i \in \mathbb{N}_0}$ using the initial condition $B_0 = \mathcal{X}_R$ and, for all $i \in \mathbb{N}$, the recursive relation

$$B_i = B_{i-1} \cup \{z \in \mathcal{X} \setminus B_{i-1} : [\underline{Q}\mathbb{I}_{B_{i-1}}](z) > 0\}.$$

Let n' be the first index such that $B_{n'} = B_{n'+1}$, then $B_{n'} = \mathcal{X}$ and $n' \leq |\mathcal{X} \setminus \mathcal{X}_R| = n \leq k$. Manipulating the recursive relation yields

$$\begin{aligned} B_i &= B_{i-1} \cup \{z \in \mathcal{X} \setminus B_{i-1} : \delta_i [\underline{Q}\mathbb{I}_{B_{i-1}}](z) > 0\} \\ &= B_{i-1} \cup \{z \in \mathcal{X} \setminus B_{i-1} : \mathbb{I}_{B_{i-1}}(z) + \delta_i [\underline{Q}\mathbb{I}_{B_{i-1}}](z) > 0\} \\ &= B_{i-1} \cup \{z \in \mathcal{X} \setminus B_{i-1} : [(I + \delta_i \underline{Q})\mathbb{I}_{B_{i-1}}](z) > 0\}. \end{aligned}$$

Fix some $x_{i-1} \in B_{i-1}$; then $\mathbb{I}_{B_{i-1}} \geq \mathbb{I}_{x_{i-1}}$. As by (R7) we know that $[\underline{Q}\mathbb{I}_{x_{i-1}}](x_{i-1}) \geq -\|\underline{Q}\|/2$, it follows by assumption on δ_i that $[(I + \delta_i \underline{Q})\mathbb{I}_{x_{i-1}}](x_{i-1}) > 0$. Using (L6), we conclude that $[(I + \delta_i \underline{Q})\mathbb{I}_{B_{i-1}}](x_{i-1}) \geq [(I + \delta_i \underline{Q})\mathbb{I}_{x_{i-1}}](x_{i-1}) > 0$. Hence

$$B_i = \{z \in \mathcal{X} : [(I + \delta_i \underline{Q})\mathbb{I}_{B_{i-1}}](z) > 0\}.$$

The sequence $\mathcal{X}_R = B_0, \dots, B_k = \mathcal{X}$ we have just constructed is equal to the sequence of Lemma 43 such that we may conclude that $\underline{T}_{1:k}\mathbb{I}_{\mathcal{X}_R} > 0$. From (L6), we now infer from $\min \underline{T}_{1:k}\mathbb{I}_A = 0$ that $\mathcal{X}_R \not\subseteq A$.

Next, we observe that

$$\bar{T}_{1:k}\mathbb{I}_A = 1 + \bar{T}_{1:k}(\mathbb{I}_A - 1) = 1 + \bar{T}_{1:k}(-\mathbb{I}_{A^c}) = 1 - \underline{T}_{1:k}\mathbb{I}_{A^c},$$

where the first equality follows from (L5) and the third equality follows from the conjugacy of $\underline{T}_{1:k}$ and $\bar{T}_{1:k}$. As we just proved that $\mathcal{X}_R \not\subseteq A$, it must hold that $\mathcal{X}_R \subseteq A^c$. By (L6), $\mathbb{I}_{A^c} \geq \mathbb{I}_{\mathcal{X}_R}$ implies that

$$\underline{T}_{1:k}\mathbb{I}_{A^c} \geq \underline{T}_{1:k}\mathbb{I}_{\mathcal{X}_R} \Leftrightarrow 1 - \underline{T}_{1:k}\mathbb{I}_{A^c} \leq 1 - \underline{T}_{1:k}\mathbb{I}_{\mathcal{X}_R}.$$

Recall that we previously proved that $\underline{T}_{1:k}\mathbb{I}_{\mathcal{X}_R} > 0$, such that $\bar{T}_{1:k}\mathbb{I}_A \leq 1 - \underline{T}_{1:k}\mathbb{I}_{\mathcal{X}_R} < 1$. This contradicts the fact that $\max \bar{T}_{1:k}\mathbb{I}_A = 1$, thereby completing the proof. ■

If the lower transition operator is linear, then the lower and upper bounds of Theorem 10 are equal. Moreover, from this special case we can immediately verify that the ergodic coefficient we use is a proper generalisation of an ergodic coefficient—the delta coefficient δ of Anderson (1991), which is equivalent to τ_1 , one of the proper coefficients of ergodicity discussed by Seneta (1981)—used in the study of precise Markov chains.

Corollary 49. *Let T be a transition matrix, then*

$$\rho(T) = \max \left\{ \frac{1}{2} \sum_{z \in \mathcal{X}} |T(x, z) - T(y, z)| : x, y \in \mathcal{X} \right\}.$$

Proof. For a transition matrix, the the upper bound of Eqn. (5) and the lower bound of Eqn. (6) in Theorem 10 are equal. Therefore

$$\begin{aligned} \rho(T) &= \max \{ \max \{ [T\mathbb{I}_A](x) - [T\mathbb{I}_A](y) : x, y \in \mathcal{X} \} : \emptyset \neq A \subset \mathcal{X} \} \\ &= \max \left\{ \max \left\{ \frac{1}{2} [T(2\mathbb{I}_A)](x) - \frac{1}{2} [T(2\mathbb{I}_A)](y) : x, y \in \mathcal{X} \right\} : \emptyset \neq A \subset \mathcal{X} \right\} \\ &= \max \left\{ \max \left\{ \frac{1}{2} [T(2\mathbb{I}_A - 1)](x) - [T(2\mathbb{I}_A - 1)](y) : x, y \in \mathcal{X} \right\} : \emptyset \neq A \subset \mathcal{X} \right\}, \end{aligned}$$

where the first equality follows from Theorem 10, the second equality follows from (L3) and the third equality follows from (L5). From the linearity of T , it follows that $[Tf](x) = \sum_{z \in \mathcal{X}} f(z)[T\mathbb{I}_z](x) = \sum_{z \in \mathcal{X}} f(z)T(x, z)$, such that

$$\begin{aligned} \rho(T) &= \max \left\{ \max \left\{ \frac{1}{2} \sum_{z \in \mathcal{X}} [2\mathbb{I}_A - 1](z) (T(x, z) - T(y, z)) : x, y \in \mathcal{X} \right\} : \emptyset \neq A \subset \mathcal{X} \right\} \\ &= \max \left\{ \max \left\{ \frac{1}{2} \sum_{z \in \mathcal{X}} [2\mathbb{I}_A - 1](z) (T(x, z) - T(y, z)) : \emptyset \neq A \subset \mathcal{X} \right\} : x, y \in \mathcal{X} \right\}. \end{aligned}$$

Solving the inner maximisation problem for some fixed $x, y \in \mathcal{X}$ is trivial: the maximising A is $\{z \in \mathcal{X} : T(x, z) \geq T(y, z)\}$ as for all $z \in \mathcal{X}$, $[2\mathbb{I}_A - 1](z)$ is 1 if $z \in A$ or -1 if $z \notin A$. This results in

$$\rho(T) = \max \left\{ \frac{1}{2} \sum_{z \in \mathcal{X}} |T(x, z) - T(y, z)| : x, y \in \mathcal{X} \right\},$$

which proves that $\rho(T)$ is indeed equal to $\delta(T)$ of Anderson (1991) or $\tau_1(T)$ of Seneta (1981). ■

Linear transition operators are not the only lower transition operators for which the lower bound of Theorem 10 is the actual value of the coefficient of ergodicity. Škulj and Hable (2013) show that this is also the case for lower transition operators defined using Choquet integrals. Let $\{L_x\}_{x \in \mathcal{X}}$ be a family of Choquet capacities, and assume that for all $x \in \mathcal{X}$, $[\underline{T} \cdot](x)$ is the Choquet integral with respect to L_x . By (Škulj and Hable, 2013, Corollary 23),

$$\rho(\underline{T}) = \max \left\{ \max \{L_x(A) - L_y(A) : x, y \in \mathcal{X}\} : 0 \neq A \subset \mathcal{X} \right\}. \quad (23)$$

This result allows us to exactly compute $\rho(\underline{T})$. However, we are often interested in $\rho(\underline{T}^k)$, where $k > 1$ is an integer. Let $k \in \mathbb{N}$ and $x \in \mathcal{X}$, then we define the Choquet capacity L_x^k for all $A \subseteq \mathcal{X}$ as $L_x^k(A) := [\underline{T}^k \mathbb{I}_A](x)$. In general, the coherent lower expectation $[\underline{T}^k \cdot](x)$ is *not* a Choquet integral with respect to the Choquet capacity L_x^k , a fact that is seemingly overlooked in (Škulj and Hable, 2013, Section 5.5). What is definitely true is that

$$\max \{ \max \{L_x^k(A) - L_y^k(A) : x, y \in \mathcal{X}\} : \emptyset \neq A \subset \mathcal{X} \}$$

is a lower bound of $\rho(\underline{T}^k)$, as it is equal to the lower bound of Theorem 10.

Lemma 50. *Let \underline{Q} be a lower transition rate operator and assume that f is an element of $\mathcal{L}(\mathcal{X})$ such that $\underline{T}_\infty f := \lim_{t \rightarrow \infty} \underline{T}_t f$ is a constant function. We let $t \in \mathbb{R}_{\geq 0}$, $\epsilon \in \mathbb{R}_{> 0}$ and $\delta_1, \dots, \delta_k \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^k \delta_i = t$ and for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| \leq 2$, and define $g := \Phi(\delta_1, \dots, \delta_k)f$. If $\|\underline{T}_t f - g\| \leq \epsilon$ and $\|g\|_c \leq \epsilon$, then*

$$\|\underline{T}_\infty f - \tilde{g}\| \leq 2\epsilon$$

and for all $\Delta \in \mathbb{R}_{\geq 0}$,

$$\|\underline{T}_{t+\Delta} f - \tilde{g}\| \leq 2\epsilon,$$

where $\tilde{g} := (\max \Phi(\delta_1, \dots, \delta_k)f + \min \Phi(\delta_1, \dots, \delta_k)f)/2$.

Proof. Note that by (L4),

$$\min \underline{T}_t f \leq \min \underline{T}_{t+\Delta} f \leq \underline{T}_\infty f \leq \max \underline{T}_{t+\Delta} f \leq \underline{T}_t f.$$

If we let $g := \Phi(\delta_1, \dots, \delta_k)f$ and assume that $\|\underline{T}_t f - g\| \leq \epsilon$, then

$$\min g - \epsilon \leq \min \underline{T}_t f \leq \min \underline{T}_{t+\Delta} f \leq \underline{T}_\infty f \leq \max \underline{T}_{t+\Delta} f \leq \underline{T}_t f \leq \max g + \epsilon.$$

Hence,

$$\underline{T}_\infty f - \tilde{g} = \underline{T}_\infty f - \max g + \frac{\max g - \min g}{2} \leq \epsilon + \|g\|_c,$$

and

$$\underline{T}_\infty f - \tilde{g} = \underline{T}_\infty f - \min g - \frac{\max g - \min g}{2} \geq -\epsilon - \|g\|_c,$$

where $\tilde{g} := (\max g + \min g)/2$. Therefore, if $\|g\|_c \leq \epsilon$, then

$$\|\underline{T}_\infty f - \tilde{g}\| \leq 2\epsilon,$$

which proves the first inequality of the statement. The proof of the second inequality of the statement is almost entirely similar. ■

Proof of Proposition 12. If \underline{Q} is ergodic, then by definition $\lim_{t \rightarrow \infty} \underline{T}_t f$ is a constant function for all $f \in \mathcal{L}(\mathcal{X})$. Therefore, the stated follows immediately from Lemma 50. \blacksquare

In Example 4, we have observed that keeping track of ϵ' increases the duration of the computations. The following proposition shows that, even if one is not really interested in the value of ϵ' , there is still a reason why one nevertheless would want to keep track of ϵ' : it could be that we can stop the approximation because we have already attained the desired maximal error.

Proposition 51. *Let \underline{Q} be a lower transition rate operator, $f \in \mathcal{L}(\mathcal{X})$ and $t, \epsilon \in \mathbb{R}_{>0}$. Let s denote some sequence $\delta_1, \dots, \delta_k$ in $\mathbb{R}_{\geq 0}$ such that $t' := \sum_{i=1}^k \delta_i \leq t$ and, for all $i \in \{1, \dots, k\}$, $\delta_i \|\underline{Q}\| \leq 2$. If $\epsilon' \leq \epsilon$ is an upper bound for $\|\underline{T}_{t'} f - \Phi(s)f\|$ and $\|\Phi(s)f\|_v \leq \epsilon - \epsilon'$, then*

$$\|\underline{T}_t f - \Phi(s)f\| \leq \epsilon.$$

Proof. First, note that by the semi-group property $\underline{T}_t f = \underline{T}_{t-t'} \underline{T}_{t'} f$. Using (L4) yields

$$\min \underline{T}_{t'} f \leq \underline{T}_t f \leq \max \underline{T}_{t'} f.$$

Hence

$$\begin{aligned} \|\underline{T}_t f - \Phi(s)f\| &= \max\{|\underline{T}_t f(x) - [\Phi(s)f](x)| : x \in \mathcal{X}\} \\ &\leq \max\{\max\{|\max \underline{T}_{t'} f - [\Phi(s)f](x)|, |\min \underline{T}_{t'} f - [\Phi(s)f](x)|\} : x \in \mathcal{X}\}, \end{aligned}$$

where the inequality follows from the obtained bounds on $\underline{T}_t f$. Let $x^+ \in \mathcal{X}$ such that $[\underline{T}_{t'} f](x^+) = \max \underline{T}_{t'} f$. Then for all $x \in \mathcal{X}$,

$$\begin{aligned} |\max \underline{T}_{t'} f - [\Phi(s)f](x)| &= |[\underline{T}_{t'} f](x^+) - [\Phi(s)f](x) - [\Phi(s)f](x^+) + [\Phi(s)f](x^+)| \\ &\leq |[\underline{T}_{t'} f](x^+) - [\Phi(s)f](x^+)| + |[\Phi(s)f](x) - [\Phi(s)f](x^+)| \\ &\leq \|\underline{T}_{t'} f - \Phi(s)f\| + \|\Phi(s)f\|_v. \end{aligned}$$

Similarly,

$$|\min \underline{T}_{t'} f - [\Phi(s)f](x)| \leq \|\underline{T}_{t'} f - \Phi(s)f\| + \|\Phi(s)f\|_v.$$

Therefore,

$$\begin{aligned} \|\underline{T}_t f - \Phi(s)f\| &\leq \max\{\max\{|\max \underline{T}_{t'} f - [\Phi(s)f](x)|, |\min \underline{T}_{t'} f - [\Phi(s)f](x)|\} : x \in \mathcal{X}\} \\ &\leq \max\{\|\underline{T}_{t'} f - \Phi(s)f\| + \|\Phi(s)f\|_v : x \in \mathcal{X}\} \\ &= \|\underline{T}_{t'} f - \Phi(s)f\| + \|\Phi(s)f\|_v. \end{aligned}$$

If we now assume that $\|\underline{T}_{t'} f - \Phi(s)f\| \leq \epsilon' \leq \epsilon$ and $\|\Phi(s)f\|_v \leq \epsilon - \epsilon'$, then

$$\|\underline{T}_t f - \Phi(s)f\| \leq \epsilon. \quad \blacksquare$$